# 4+1 Formalism for a Local Metric with Parameterized Evolution


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# Stueckelberg-Horwitz-Piron (SHP) Formalism

| **Electrodynamics** | | **Relativity** |
| --- | --- | --- |
| Maxwell's equations | ⟶ Einstein | Lorentz group |
| Manifest covariance | ⟵ Minkowski & Fock | Tensor formulation |
| $\partial_\nu F^{\mu\nu}(x) = eJ^\mu(x)$ | | $\dot{x}^\mu = dx^\mu/ds$ |
| $M\ddot{x}^\mu = eF^{\mu\nu}(x)\dot{x}_\nu$ | | $ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$ |
| Pair creation/annihilation | ⟶ Stueckelberg | External evolution parameter |
| $E = p^0 = M\dot{x}^0 < 0$ | | $\dot{x}^\mu = dx^\mu/d\tau$ <br> $ds^2 = -\dot{x}^2(\tau)d\tau^2$ |
| Covariant canonical dynamics | ⟵ Horwitz & Piron | Unconstrained phase space |
| $K = \frac{[p - eA(x)]^2}{2M} + V(x)$ | | $\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$ |
| $\tau$-dependent gauge field | ⟶ Horwitz, Sa'ad, Arshansky, Land | Evolving block universe |
| $A_\mu(x) \longrightarrow a_\mu(x,\tau)$ | | $\mathcal{M} \to \mathcal{M}(\tau)$ |
| $V(x) \longrightarrow -ea_5(x,\tau)$ | | $\frac{d}{d\tau}$ (particle + field masses) $= 0$ |

# Stueckelberg-Horwitz-Piron (SHP) Formalism

Covariant canonical mechanics with parameterized evolution

8D unconstrained phase space $\implies$ $\tau \neq$ proper time

$$x^\mu(\tau),\ \dot{x}^\mu(\tau) \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \lambda, \mu, \nu, \ldots = 0, 1, 2, 3$$

Canonical electrodynamics with scalar Hamiltonian ($K =$ total mass)

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e\, \dot{x}^\mu a_\mu(x,\tau) + e\, a_5(x,\tau) \qquad 0 = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}$$

$$K = \frac{[p - ea(x,\tau)]^2}{2M} - ea_5(x,\tau) \qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

Extend SHP to include pseudo-5D metric describing $\tau$-evolution of spacetime

$$L = \frac{1}{2} M g_{\alpha\beta}(x, \tau) \dot{x}^\alpha \dot{x}^\beta \qquad \alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 5 \qquad x^5 = c_5 \tau$$

# SHP — Geometry and Evolution

4D block universe $\mathcal{M}(\tau)$ at each $\tau$

## Physical event $x^\mu(\tau)$ in SHP

Irreversible occurrence **at** time $\tau$

$$\tau_2 > \tau_1 \implies \begin{cases} x^\mu(\tau_2) \text{ occurs } \textbf{after } x^\mu(\tau_1) \\ x^\mu(\tau_2) \textbf{ cannot change } x^\mu(\tau_1) \\ \text{No grandfather paradox} \end{cases}$$

## Evolution

4D block universe $\mathcal{M}(\tau)$ **occurs** at $\tau$

Infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occurs at $\tau + d\tau$

$$\mathcal{M}(\tau) \xrightarrow[\text{Hamiltonian } K \text{ generates evolution in } \tau]{} \mathcal{M}(\tau + d\tau)$$

$$\left.\begin{array}{r} \text{scalar } K \\ \text{external } \tau \end{array}\right\} \implies \text{No conflict with general diffeomorphism invariance}$$

# Geometry and Trajectory

## Standard approach to motion in general relativity

Two neighboring events in spacetime manifold $\mathcal{M}$ (instantaneous displacement)

Interval $\delta x^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu = (x_2 - x_1)^2$

Invariance of interval — geometrical statement about $\mathcal{M}$

Trajectory

Map arbitrary parameter $\zeta$ to sequence of events $x^\mu(\zeta)$

Timelike interval between any two events $\Rightarrow$ take $\zeta \to s$ (proper time)

In 4D block universe $\mathcal{M}$

Trajectory — sequence of instantaneous timelike displacements

"Motion" appears as displacements in $x^0(s)$

Path length $\longrightarrow$ Lagrangian

$$\delta x^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \delta s^2 \qquad\qquad g = (-,+,+,+)$$

$$L_{\text{constrained}} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \qquad\qquad L_{\text{unconstrained}} = \frac{1}{2}\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

Two neighboring events:

$$x^\mu(\tau) \in \mathcal{M}(\tau) \qquad \bar{x}^\mu(\tau + \delta\tau) \in \mathcal{M}(\tau + \delta\tau)$$

Distance

$$dx^\mu = \bar{x}^\mu(\tau + \delta\tau) - x^\mu(\tau) \simeq \bar{x}^\mu(\tau) + \dot{\bar{x}}^\mu(\tau)\delta\tau - x^\mu(\tau) = \delta x^\mu + \dot{\bar{x}}^\mu \delta\tau$$

Squared interval (referred to $x$ coordinates)

$$dx^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu + g_{\mu\nu}\dot{\bar{x}}^\mu \delta x^\nu \delta\tau + g_{\mu\nu}\dot{\bar{x}}^\mu \dot{\bar{x}}^\nu \delta\tau^2 \simeq \underbrace{g_{\alpha\beta}(x,\tau)\,\delta x^\alpha \delta x^\beta}_{\alpha,\beta=0,1,2,3,5}$$

Contributions to interval

$g_{\mu\nu}\ \delta x^\mu \delta x^\nu$

- Geometrical interval between two events at $\tau$
- Expresses symmetries of spacetime manifold $\mathcal{M}$

$g_{55}\ \delta x^5 \delta x^5$

- Dynamical interval between events in $\mathcal{M}(\tau)$ and $\mathcal{M}(\tau + \delta\tau)$
- Expresses symmetries of evolution generated by Hamiltonian $K$

# Example in space

Particle in 2D space — expanding disk with radius $R(\tau) = \frac{1}{2}g\tau^2$

Points on expanding disk

$$\mathbf{q} = R(\tau)(\cos\theta, \sin\theta) \qquad \bar{\mathbf{q}} = \bar{R}(\tau + \delta\tau)\left(\cos\bar{\theta}, \sin\bar{\theta}\right)$$

Distance

$$d\mathbf{q} = \bar{\mathbf{q}} - \mathbf{q} \approx (\delta R + \delta_\tau R(\tau))\,\hat{\mathbf{R}} + R\delta\theta\ \hat{\boldsymbol{\theta}}$$

Geometrical distances: $\delta\theta = \bar{\theta} - \theta \qquad \delta R = \bar{R}(\tau) - R(\tau)$

Dynamical distance: $\delta_\tau R(\tau) = R(\tau + \delta\tau) - R(\tau) = g\tau\delta\tau$

Interval

$$d\mathbf{q}^2 = \delta R^2 + R^2\delta\theta^2 + g^2\tau^2\delta\tau^2 + 2g\tau\delta R\delta\tau = g_{ab}\delta\zeta^a\delta\zeta^b$$

Pseudo-3D metric

$$\delta\zeta = (\delta R, \delta\theta, \delta\tau) \qquad g_{ab} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & R^2 & 0 \\ 1 & 0 & g\tau \end{pmatrix}$$

# Example in space

## Equations of motion

### Lagrangian

$$L = \frac{1}{2} M g_{ab} \dot{\zeta}^a \dot{\zeta}^b = \frac{1}{2} M \left( \dot{R}^2 + R^2 \dot{\theta}^2 + 2 g \tau \dot{R} + g^2 \tau^2 \right)$$

$$\left. \begin{array}{r} 0 = \dfrac{d}{d\tau} M \left( \dot{R} + g\tau \right) - M R \dot{\theta}^2 \\ 0 = \dfrac{d}{d\tau} \left( M R^2 \dot{\theta} \right) \longrightarrow M R^2 \dot{\theta} = \ell \end{array} \right\} \longrightarrow M \ddot{R} = \frac{\ell^2}{M R^3} - M g$$

### Qualitative result

Particle at edge of disk sees force $F = \ell^2 / M R^3 - M g$

$M g > \ell^2 / M R^3 \implies$ Particle moves at edge of disk
As if attracted by gravitational force

$F_R = -M g \longrightarrow$ Appears as “external” force
Enters through evolution of circular geometry

# Canonical Mechanics in General 5D Spacetime

Lagrangian

$$L = \frac{1}{2} M g_{\alpha\beta}(x^\mu, x^5)\dot{x}^\alpha \dot{x}^\beta \qquad \lambda, \mu, \nu = 0, 1, 2, 3 \qquad \alpha, \beta, \gamma = 0, 1, 2, 3, 5$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha}\right)$$

Canonical momentum

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = M g_{\alpha\beta}\dot{x}^\beta \qquad \longrightarrow \qquad \dot{x}^\alpha = \frac{1}{M} g^{\alpha\beta} p_\beta$$

Conserved Hamiltonian

$$K = \dot{x}^\alpha p_\alpha - L = \frac{1}{2M} g^{\alpha\beta} p_\alpha p_\beta \qquad \frac{dK}{d\tau} = M g_{\alpha\beta}\dot{x}^\alpha \frac{D\dot{x}^\beta}{D\tau} = 0$$

Poisson bracket

$$\frac{dK}{d\tau} = \{K, K\} + \frac{\partial K}{\partial \tau} = \frac{1}{2M} p_\alpha p_\beta \frac{\partial g^{\alpha\beta}}{\partial \tau} = 0$$

# Break 5D symmetry $\longrightarrow$ 4D+1

Constrain non-dynamical scalar $x^5 \equiv c_5 \tau$

$$L = \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} M g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu + M c_5\, g_{\mu 5} \dot{x}^\mu + \frac{1}{2} M c_5^2\, g_{55}$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\alpha}{D\tau} = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma \longrightarrow \begin{cases} \ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma} \dot{x}^\lambda \dot{x}^\sigma + 2 c_5 \Gamma^\mu_{5\sigma} \dot{x}^\sigma + c_5^2 \Gamma^\mu_{55} = 0 \\ \ddot{x}^5 = \dot{c}_5 \equiv 0 \end{cases}$$

Symmetry-broken connection

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left( \partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta} \right) \qquad\qquad \Gamma^5_{\alpha\beta} \equiv 0$$

Hamiltonian

$$K = p_\mu \dot{x}^\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu - M c_5\, g_{\mu 5} \dot{x}^\mu - \frac{1}{2} M c_5^2\, g_{55}$$

$$\frac{dK}{d\tau} = \frac{\partial K}{\partial \tau} = -\frac{1}{2} M \dot{x}^\mu \dot{x}^\nu\, \partial_\tau g_{\mu\nu} - M c_5 \dot{x}^\mu\, \partial_\tau g_{\mu 5} - \frac{1}{2} M c_5^2\, \partial_\tau g_{55}$$

# Matter

## Non-thermodynamic dust

Number of events per spacetime volume $= n(x,\tau)$

Particle mass density $= \rho(x,\tau) = Mn(x,\tau)$

5-component event current $= j^\alpha(x,\tau) = \rho(x,\tau)\dot{x}^\alpha(\tau) = Mn(x,\tau)\dot{x}^\alpha(\tau)$

Continuity equation

$$\nabla_\alpha j^\alpha = \nabla_\mu j^\mu + \partial_\tau \rho$$

Mass-energy-momentum tensor

$$\nabla_\beta T^{\alpha\beta} = 0 \qquad\qquad T^{\alpha\beta} = \rho\dot{x}^\alpha\dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = \rho\dot{x}^\mu\dot{x}^\nu \\ T^{5\beta} = c_5 j^\beta \end{cases}$$

## Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$$

# Weak Field Approximation

Small perturbation to flat metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \left(h_{\alpha\beta}\right)^2 \approx 0 \qquad h \simeq \eta^{\alpha\beta} h_{\alpha\beta}$$

Define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h \longrightarrow$ Einstein equations

$$\frac{16\pi G}{c^4} T_{\alpha\beta} = \partial_\beta \partial_\gamma \bar{h}^\gamma_\alpha + \partial_\alpha \partial_\gamma \bar{h}^\gamma_\beta - \partial_\gamma \partial_\gamma \bar{h}_{\alpha\beta} - \partial_\alpha \partial_\beta \bar{h}$$

Impose gauge condition $\partial_\lambda \bar{h}^{\alpha\lambda} = 0 \longrightarrow$ wave equation

$$\frac{16\pi G}{c^4} T_{\alpha\beta} = -\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} = -\left(\partial^\mu \partial_\mu + \frac{\eta_{55}}{c_5^2}\partial_\tau^2\right)\bar{h}_{\alpha\beta}$$

Solve with principal part Green's function

$$\bar{h}_{\alpha\beta}(x,\tau) = \frac{4G}{c^4}\int d^3x' \, \frac{T_{\alpha\beta}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau\right)}{|\mathbf{x}-\mathbf{x}'|}$$

# Post-Newtonian Approximation

Point source $X = (cT(\tau), \mathbf{0})$ in co-moving frame where $\dot{T} = 1 + \alpha(\tau)/2$ and $\alpha^2 \approx 0$

$$T^{00} = mc^2 \dot{T}^2 \delta^3(\mathbf{x}) \rho(t - T(\tau)) \qquad T^{\alpha i} = 0 \qquad T^{55} = \frac{c_5^2}{c^2} T^{00} \approx 0$$

Writing $M(\tau) = m\rho(t - T(\tau))$ a slowly varying density function

$$\bar{h}^{00}(x,\tau) = \frac{4GM}{c^2 R} \dot{T}^2 \qquad \bar{h}^{\alpha i}(x,\tau) = 0 \qquad \bar{h}^{55}(x,\tau) = \frac{c_5^2}{c^2} \bar{h}^{00} \approx 0$$

$$\Gamma^{\mu}_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} \qquad \Gamma^{\mu}_{50} = \frac{1}{2c_5} \eta^{\mu 0} \partial_\tau h_{00}$$

Equations of motion for a test particle in spherical coordinates putting $\theta = \pi/2$

$$\ddot{t} = (\partial_\tau h_{00}) \dot{t} + \dot{\mathbf{x}} \cdot (\nabla h_{00}) \dot{t}^2 \approx \frac{2GM}{c^2 R} \left(1 + \frac{\alpha(\tau)}{2}\right) \dot{\alpha}(\tau) \dot{t}$$

$$\ddot{\mathbf{x}} = \frac{c^2}{2} (\nabla h_{00}) \dot{t}^2 \longrightarrow \begin{cases} 2\dot{R}\dot{\phi} + R\ddot{\phi} = 0 \longrightarrow \dot{\phi} = \dfrac{L}{MR^2} \\ \ddot{R} - \dfrac{L^2}{M^2 R^3} = -\dfrac{GM}{R^2} \dot{t}^2 \dot{T}^2 \end{cases}$$

# Post-Newtonian Evolution

Solution to $t$ equation neglecting $\dot{R}/c \ll 1$ and $\partial_\tau \rho \approx 0$

$$\dot{t} = \exp\left[\frac{2GM}{c^2 R}\left(\alpha + \frac{1}{4}\alpha^2\right)\right] \longrightarrow \dot{t}^2 \dot{T}^2 \simeq 1 + \frac{1}{2}\left(1 + \frac{2GM}{c^2 R}\right)\alpha$$

Taking $\alpha = 0$ recovers $\dot{t} = 1$ for Newtonian case

Using solution to $t$ equation in radial equation

$$\frac{d}{d\tau}\left\{\frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{L^2}{M^2 R^2} - \frac{GM}{R}\left(1 + \frac{1}{2}\alpha(\tau)\right)\right\} = -\frac{GM}{2R}\frac{d}{d\tau}\alpha(\tau)$$

LHS is $\dfrac{dK}{d\tau}$ where $K$ is particle Hamiltonian $=$ particle mass

In source rest frame, $\dot{T} \neq 1 \Rightarrow \Delta$energy without $\Delta$momentum $\Rightarrow \Delta$mass

Source transfers mass to perturbed metric field $h_{\alpha\beta}$

Particle mass not generally conserved $\longrightarrow$ particle absorbs mass from $h_{\alpha\beta}$

Taking $\alpha = 0$ recovers nonrelativistic motion with conserved Hamiltonian

# 3+1 Formalism in General Relativity

Time evolution formalism for Einstein equations

- Formulate field equations as initial value problem
- Decompose 10 components of spacetime metric into:
  - 6 component space metric
  - 3 component (spatial) shift vector
  - 1 component (time) lapse
- Decompose 10 components of Einstein equations into:
  - 6 component partial differential equation (PDE) of second order in $\partial_t$
  - 4 constraints on space metric, its 1st $t$-derivative, energy-momentum

Based on mathematics of embedded hypersurfaces

- Darmois (1927), Lichnerowicz (1939), Choquet-Bruhat (1952)
  - Intrinsic structure of hypersurface
  - Extrinsic structure imposed by embedding hypersurface in larger manifold
- Arnowitt, Deser, Misner (1962)
  - ADM Hamiltonian formulation of GR
  - Intrinsic and extrinsic structure $\longrightarrow$ canonical conjugate field variables

4+1 formalism generalizes 3+1 framework as presented in:

- Gourgoulhon (2007), Bertschinger (2005), ADM (1962), Isham (1992), Blau (2020)

# Schematic Outline of 3+1 Formalism

Define a foliation of 4D spacetime

- Choose scalar field $t(x)$ on spacetime
- Hypersurface = simultaneous spacetime points $x$ sharing $t(x) = constant$
- Choose $t(x)$ to identify spacelike hypersurfaces (3D space manifold at time $t$)
  - Each vector $V$ tangent to each hypersurface is spacelike ($V^2 > 0$ )
  - Vector $n$ normal to hypersurface is timelike ($n^2 = -1$ )

Projection operators split tangent and normal components of 4D objects

- Induced space metric $\gamma_{ij}$ is spacelike projection of spacetime metric $g_{\mu\nu}$
- Spacetime metric $g_{\mu\nu} \longrightarrow \{\gamma_{ij}$, (normal) lapse $N$, and (tangent) shift $N^i\}$
- Spacelike projection of 4D covariant derivative compatible with $\gamma_{ij}$
- Spacelike projection of 4D curvature $\longrightarrow$ usual 3D (intrinsic) curvature
- Timelike projection of 4D curvature $\longrightarrow$ extrinsic curvature of $t$-evolving space

Einstein equations

- Project Einstein equations onto spacelike hypersurface and timelike normal
- 10 components of Einstein equations split into two groups
  - 6 second order PDEs describing $t$-evolution of space metric $\gamma_{ij}$
  - 4 non-evolving constraints on initial conditions

# 4+1 Formalism in SHP General Relativity

## Schematic Outline I: The Embedding

5D pseudo-spacetime coordinates $X = (x, c_5\tau) \in \mathcal{M}_5 = \mathcal{M} \times R$

- Manifold $\mathcal{M}_5$ an admixture of 4D spacetime geometry and $\tau$-evolution
- In flat pseudo-spacetime $g_{\alpha\beta} \to \eta_{\alpha\beta} = \text{diag}\,(-1,1,1,1,\sigma)$ where $\sigma = \pm 1$

Natural foliation of 5D pseudo-spacetime $\mathcal{M}_5 \longrightarrow$ spacetime hypersurface $\sum_\tau$

- Time function $\tau(X) \longrightarrow \sum_{\tau_0} = \left\{X \in \mathcal{M}_5 \mid S(X) = \tau(X) - \tau_0 = 0\right\}$
- Vector $n$ normal to hypersurface $\sum_{\tau_0}$ is $\tau$-like ($n^2 = \sigma = \pm 1$ )

Projection operators split tangent and normal components of 5D objects

- Induced spacetime metric $\gamma_{\mu\nu}$ is projection onto $\mathcal{M}$ of 5D metric $g_{\alpha\beta}$
- 5D metric $g_{\alpha\beta} \longrightarrow \left\{\gamma_{\mu\nu}\text{, lapse } N\text{, and (tangent) shift } N^\mu\right\}$
- Spacetime projection of 5D covariant derivative compatible with $\gamma_{\mu\nu}$
- Spacetime projection of 5D curvature $\longrightarrow$ usual 4D (intrinsic) curvature $R_{\mu\nu\lambda\rho}$
- $\tau$ projection of 5D curvature $\longrightarrow$ extrinsic curvature of $\tau$-evolution $K_{\alpha\beta}$

5D Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = 0$$

5 constraints on solution to field equations

Project Einstein equations onto spacetime hypersurface and $\tau$-like normal

Decompose 15 components of field equations into two groups

10 unconstrained second order PDEs

Describe $\tau$-evolution of spacetime metric $\gamma_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$

5 non-evolving constraints on initial conditions: $\left\{ \gamma_{\mu\nu} , \partial_\tau \gamma_{\mu\nu} , T_{\alpha\beta} \right\}$

# Foliation of 5D Pseudo-Spacetime $\mathcal{M}_5 = \mathcal{M} \times R$

## Embedding

Injective mapping $\Phi : \mathcal{M} \to \mathcal{M}_5$ with $\begin{cases} x^\mu \in \mathcal{M},\ \mu = 0,1,2,3 \\ X^\alpha = (x, c_5\tau) \in \mathcal{M}_5,\ \alpha = 0,1,2,3,5 \end{cases}$

## 4D hypersurface — implicitly defined submanifold

$$\sum_{\tau_0} = \left\{ X \in \mathcal{M}_5 \;\middle|\; S(X) = 0 \right\} \text{ where } S(X) = \tau(X) - \tau_0 = X^5/c_5 - \tau_0$$

## Rank 4 Jacobian

$$E^\alpha_\mu = \left( \frac{\partial X^\alpha}{\partial x^\mu} \right)_{\tau_0} \longrightarrow E_\mu = \partial_\mu = \partial/\partial x^\mu \ \text{ as basis for tangent space of } \sum_{\tau_0}$$

## Unit normal to $\sum_{\tau_0}$

$$n_\alpha = \sigma \left| g^{55} \right|^{-1/2} \partial_\alpha S\left(X\right) \longrightarrow \begin{cases} n \cdot E_\mu = n_\alpha E^\alpha_\mu = 0 \\ n^2 = g^{\alpha\beta} n_\alpha n_\beta = \sigma \end{cases}$$

## Induced metric on $\mathcal{M}$

$$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta = g_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X^\beta}{\partial x^\nu} dx^\mu dx^\nu = \gamma_{\mu\nu} dx^\mu dx^\nu \longrightarrow \gamma_{\mu\nu} = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu$$

# Projection Operators

## Projections

$$A \in \mathcal{T}_x\left(\mathcal{M}_5\right) = \text{tangent space of } \mathcal{M}_5 \longrightarrow \begin{cases} A_{\parallel} = \sigma\left(A \cdot n\right) n \\ A_{\perp} = A - \sigma\left(A \cdot n\right) n \end{cases}$$

## Normal Projection Operator

$$N_{\alpha\beta} = \sigma n_\alpha n_\beta \qquad N_{\alpha\gamma} N^{\gamma\beta} = \sigma^2 n^2 \; n_\alpha n^\beta = N_\alpha^\beta$$

## Tangent Projection Operator

$$P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \qquad P_\beta^\alpha = g^{\alpha\beta} - \sigma n^\alpha n^\beta \qquad P_{\alpha\gamma} P^{\gamma\beta} = P_\alpha^\beta = \delta_\alpha^\beta - \sigma n_\alpha n^\beta$$

## Completeness Relation

$$\delta_\beta^\alpha = P_\beta^\alpha + \sigma n^\alpha n_\beta$$

## Projector $P_{\mu\nu}$ restricted to $\sum_\tau$ is metric $\gamma_{\mu\nu}$

$$V \in \mathcal{T}_x\left(\mathcal{M}_5\right) \longrightarrow V_\perp^\alpha = P_\beta^\alpha V^\beta \in \mathcal{T}_x\left(\textstyle\sum_\tau\right) \longrightarrow \exists\, v \in \mathcal{T}_x\left(\mathcal{M}\right) \text{ such that } V_\perp^\alpha = v^\mu E_\mu^\alpha$$

$$\gamma_{\mu\nu} = g_{\alpha\beta} E_\mu^\alpha E_\nu^\beta = \left(P_{\alpha\beta} + \sigma n_\alpha n_\beta\right) E_\mu^\alpha E_\nu^\beta = P_{\alpha\beta} E_\mu^\alpha E_\nu^\beta = P_{\mu\nu}$$

# Decomposition of the Metric

Time evolution

$$\text{Embedding } X^\alpha(x,\tau) \longrightarrow \begin{cases} \text{hypersurface} & \sum_{\tau_0} = \{X^\alpha \mid \tau(X) = \tau_0\} \\ \text{trajectory} & X^\alpha(\tau) = \{X^\alpha(x,\tau) \mid x = x_0\} \end{cases}$$

General trajectory: normal component $\in \sum_{\tau_0+\delta\tau}$ and tangent component $\in \sum_{\tau_0}$

$$\tau \to \tau + \delta\tau \ \Rightarrow X^\alpha + \left(\frac{\partial X^\alpha}{\partial \tau}\right)_{x_0} \delta\tau = X^\alpha + \left(N n^\alpha + N^\mu E^\alpha_\mu\right)\delta\tau$$

Lapse function $N$ and Shift vector $N^\mu$

Spacetime shift

$$X^\alpha \to X^\alpha + \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)_{\tau_0} \delta x^\mu = X^\alpha + E^\alpha_\mu \delta x^\mu$$

5D interval

$$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta = g_{\alpha\beta}\left[N n^\alpha c_5 d\tau + E^\alpha_\mu\left(N^\mu c_5 d\tau + dx^\mu\right)\right]\left[N n^\beta c_5 d\tau + E^\beta_\nu\left(N^\nu c_5 d\tau + dx^\nu\right)\right]$$

Decomposition of metric using $\quad n^2 = \sigma \qquad n_\alpha E^\alpha_\mu = 0 \qquad \gamma_{\mu\nu} = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \dfrac{1}{N^2} N^\mu N^\nu & -\sigma \dfrac{1}{N^2} N^\mu \\ -\sigma \dfrac{1}{N^2} N^\mu & \sigma \dfrac{1}{N^2} \end{bmatrix}$$

# Intrinsic Geometry

## Covariant derivatives — compatibility — curvature

On $\mathcal{M}_5$ $\quad g_{\beta\gamma} \longrightarrow \Gamma_{\sigma\beta\gamma} \longrightarrow \nabla_\alpha g_{\beta\gamma} = 0 \longrightarrow \left[\nabla_\beta, \nabla_\alpha\right] X_\delta = X_\gamma R^\gamma_{\delta\alpha\beta}$

On $\mathcal{M}$ $\quad \gamma_{\mu\nu} \longrightarrow \Gamma_{\mu\nu\lambda} \longrightarrow D_\mu \gamma_{\nu\lambda} = 0 \longrightarrow \left[D_\nu, D_\mu\right] X_\rho = X_\lambda R^\lambda_{\rho\mu\nu}$

## Projected Covariant Derivative

For $V \in \mathcal{M}_5$ define $\bar{\nabla}_\alpha V^\perp_\beta = \left(P^\gamma_\alpha \nabla_\gamma\right)\left(P^\delta_\beta V_\delta\right) = P^\gamma_\alpha P^\delta_\beta \nabla_\gamma V_\delta$

using $\bar{\nabla}_\alpha P_{\beta\gamma} = -\sigma P^{\alpha'}_\alpha P^{\beta'}_\beta P^{\gamma'}_\gamma \left(\left(\nabla_{\alpha'} n_{\gamma'}\right) n_{\beta'} + n_{\beta'} \nabla_{\alpha'} n_{\gamma'}\right) = 0$

Associate $\quad D_\alpha = \bar{\nabla}_\alpha$ on $\sum_\tau \longrightarrow D_\mu$ on $\mathcal{M}$

through $\quad D_\mu = E^\alpha_\mu D_\alpha = E^\alpha_\mu P^\gamma_\alpha \nabla_\gamma = E^\gamma_\mu \nabla_\gamma$

## Projected Curvature $\bar{R}^\gamma_{\delta\alpha\beta}$

On $\sum_\tau$ $\quad \gamma_{\beta\gamma} \longrightarrow \bar{\Gamma}_{\sigma\beta\gamma} \longrightarrow D_\alpha \gamma_{\beta\gamma} = 0 \longrightarrow \left[D_\beta, D_\alpha\right] X_\delta = X_\gamma \bar{R}^\gamma_{\delta\alpha\beta}$

Curvature of $\mathcal{M}$ as manifold embedded in $\mathcal{M}_5$ as $\sum_\tau$

Gradient $\nabla_\gamma n_\delta$ in $\mathcal{M}_5$

Extrinsic curvature = projection of gradient onto $\sum_\tau \longrightarrow \ K_{\alpha\beta} = -P_\alpha^\gamma P_\beta^\delta \nabla_\gamma n_\delta$

Using $n^2 = \sigma \longrightarrow \left(\nabla_\alpha n_\beta\right) n^\beta = 0 \longrightarrow \nabla_\alpha n_\beta \in \sum_\tau \longrightarrow P_\beta^\delta \left(\nabla_\gamma n_\delta\right) = \nabla_\gamma n_\beta$

$$K_{\alpha\beta} = -P_\alpha^\gamma \nabla_\gamma n_\beta = -\nabla_\alpha n_\beta + \sigma n_\alpha \left(n^\gamma \nabla_\gamma n_\beta\right)$$

Using $n_\beta = \sigma \left|g^{55}\right|^{-1/2} \partial_\beta S\left(X\right) = \sigma N \nabla_\beta S\left(X\right)$

Expanding $n^\gamma \nabla_\gamma n_\beta = n^\gamma \left(\nabla_\gamma N\right) \dfrac{n_\beta}{N} + n^\gamma N \nabla_\beta \left(\dfrac{n_\gamma}{N}\right) = -\sigma \dfrac{1}{N} D_\beta N$

$$K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$$

## Time Evolution of $\sum_\tau$

$$\delta X^\alpha = \left(\frac{\partial X^\alpha}{\partial x^5}\right)_{x_0} \delta x^5 = \left(\frac{\partial X^\alpha}{\partial \tau}\right)_{x_0} \delta\tau \longrightarrow E_5^\alpha = (\partial_5)^\alpha = N n^\alpha + N^\mu E_\mu^\alpha$$

Define $m^\alpha = N n^\alpha \;\Rightarrow\; m^2 = \sigma N^2 \;\longrightarrow\; \partial_5 = m + \mathbf{N}$

Using $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$

$$\nabla_\beta m_\alpha = N \nabla_\beta n_\alpha + n_\alpha \nabla_\beta N = -N K_{\beta\alpha} - n_\beta D_\alpha N + n_\alpha \nabla_\beta N$$

## Lie derivative of $\gamma_{\mu\nu}$ along $\partial_5$

$$\mathcal{L}_5 = \mathcal{L}_m + \mathcal{L}_\mathbf{N}$$

Using $\mathcal{L}_m \, \gamma_{\alpha\beta} = m^\gamma \nabla_\gamma \gamma_{\alpha\beta} + \gamma_{\gamma\beta} \nabla_\alpha m^\gamma + \gamma_{\alpha\gamma} \nabla_\beta m^\gamma = -2N K_{\alpha\beta}$

Evolution equation for spacetime metric

$$\mathcal{L}_5 \, \gamma_{\alpha\beta} - \mathcal{L}_\mathbf{N} \, \gamma_{\alpha\beta} = -2N K_{\alpha\beta} \;\longrightarrow\; \mathcal{L}_5 \, \gamma_{\mu\nu} - \mathcal{L}_\mathbf{N} \, \gamma_{\mu\nu} = -2N K_{\mu\nu}$$

# Decomposition of the Riemann Tensor

## Gauss-Codazzi Relations

Using $\delta_\alpha^{\alpha'} = P_\alpha^{\alpha'} + \sigma n_\alpha n^{\alpha'}$ $\quad E_\mu^\alpha P_\alpha^{\alpha'} = E_\mu^{\alpha'}$ $\quad n_\alpha E_\mu^\alpha = 0$ to expand

$$R_{\delta\alpha\beta}^{\gamma} = \delta_\alpha^{\alpha'} \delta_\beta^{\beta'} \delta_{\gamma'}^{\gamma} \delta_\delta^{\delta'} R_{\delta'\alpha'\beta'}^{\gamma'} \longrightarrow \begin{cases} E_\mu^\alpha E_\nu^\beta E_\gamma^\lambda E_\sigma^\delta P_\alpha^{\alpha'} P_\beta^{\beta'} P_{\gamma'}^{\gamma} P_\delta^{\delta'} R_{\delta'\alpha'\beta'\gamma'} = R_{\sigma\mu\nu}^{\lambda} \\ E_\mu^\alpha E_\nu^\beta E_\gamma^\lambda P_{\gamma'}^{\gamma} n^\delta P_\alpha^{\alpha'} P_\beta^{\beta'} R_{\delta\alpha'\beta'}^{\gamma'} = \sigma N R_{5\mu\nu}^{\lambda} \\ E^{\alpha\mu} E_\nu^\beta P_{\alpha\alpha'} n^\delta P_\beta^{\beta'} n^\gamma R_{\delta\beta'\gamma}^{\alpha'} = N^2 R_{5\nu 5}^{\mu} \end{cases}$$

where $R_{\delta\alpha\beta}^{\gamma} n^\delta n^\alpha n^\beta = 0$ because of the symmetries of the Riemann tensor

Gauss relation

Decomposing projected Ricci equation for $V$ tangent to $\sum_\tau$

$$\left[D_\beta, D_\alpha\right] V_\delta = V_\gamma \bar{R}_{\delta\alpha\beta}^{\gamma} \longrightarrow R^{\mu}{}_{\nu\lambda\rho} = \bar{R}^{\mu}{}_{\nu\lambda\rho} - \sigma\left(K_\lambda^\mu K_{\rho\nu} - K_\rho^\mu K_{\lambda\nu}\right)$$

Codazzi relation

Decomposing Ricci equation for unit normal $n$

$$\left[\nabla_\beta, \nabla_\alpha\right] n^\gamma = R_{\gamma'\alpha\beta}^{\gamma} n^{\gamma'} \longrightarrow R_{\mu\nu\lambda}^{5} = \sigma \frac{1}{N}\left(D_\lambda K_{\nu\mu} - D_\nu K_{\lambda\mu}\right)$$

# Evolution of the Extrinsic Curvature

Decompose Ricci equation for unit normal $n$

$$\nabla_\beta \nabla_\gamma n^\alpha - \nabla_\gamma \nabla_\beta n^\alpha = R^\alpha_{\delta\beta\gamma} n^\delta$$

Project once onto $n$ and twice onto $\sum_\tau$

$$P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_\beta \left( \nabla_{\beta'} \nabla_{\gamma'} n^{\alpha'} - \nabla_{\gamma'} \nabla_{\beta'} n^{\alpha'} \right) = P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_\beta \; R^{\alpha'}_{\delta\beta'\gamma'} n^\delta$$

Use $K_{\alpha\beta} = -\nabla_\alpha n_\beta - n_\alpha \frac{1}{N} D_\beta N$

to obtain $P_{\alpha\alpha'} n^{\gamma'} P^{\beta'}_\beta \; R^{\alpha'}_{\delta\beta'\gamma'} n^\delta = -K_{\alpha\gamma} K^\gamma{}_\beta + \frac{1}{N} D_\beta D_\alpha N + P^\gamma{}_\alpha P^\delta{}_\beta \, n^\varepsilon \nabla_\varepsilon K_{\gamma\delta}$

Lie derivative of $K_{\alpha\beta}$

$$\left( \mathcal{L}_5 - \mathcal{L}_{\mathbf{N}} \right) K_{\alpha\beta} = \mathcal{L}_m \, K_{\alpha\beta} = m^\gamma \nabla_\gamma K_{\alpha\beta} + K_{\gamma\beta} \nabla_\alpha m^\gamma + K_{\alpha\gamma} \nabla_\beta m^\gamma$$

Using $\nabla_\beta m^\alpha = -N K^\alpha_\beta - n_\beta D^\alpha N + n^\alpha \nabla_\beta N$

and combining the above leads to

$$P^{\alpha'}{}_\alpha P^{\beta'}{}_\beta R_{\;\alpha'\beta'} = \sigma \frac{1}{N} \mathcal{L}_m \, K_{\alpha\beta} + \sigma \frac{1}{N} D_\alpha D_\beta N + \bar{R}_{\alpha\beta} - \sigma K K_{\alpha\beta} + \sigma 2 K^\delta_\alpha K_{\beta\delta}$$

# Decomposition of the Einstein Equations

## Evolution Equation for $K_{\alpha\beta}$

### Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta} \longrightarrow R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) \qquad T = g^{\alpha\beta}T_{\alpha\beta}$$

### Decompose $T_{\alpha\beta}$

$$T_{\alpha\beta} = T_{\alpha'\beta'}\left(P_{\alpha}^{\alpha'} + \sigma n^{\alpha'}n_{\alpha}\right)\left(P_{\beta}^{\beta'} + \sigma n^{\beta'}n_{\beta}\right) = S_{\alpha\beta} + 2\sigma n_{\alpha}p_{\beta} + n_{\alpha}n_{\beta}\kappa$$

where

$$S_{\alpha\beta} = P_{\alpha}^{\alpha'}P_{\beta}^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{\mu\nu} \qquad p_{\beta} = -n^{\alpha'}P_{\beta}^{\beta'}T_{\alpha'\beta'} \longleftrightarrow T_{5\mu}$$

$$\kappa = n^{\alpha}n^{\beta}T_{\alpha\beta} \longleftrightarrow T_{55} \qquad T = S + \sigma\kappa$$

Combining $P_{\alpha}^{\alpha'}P_{\beta}^{\beta'}\left(T_{\alpha'\beta'} - \frac{1}{2}g_{\alpha'\beta'}T\right) = S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\left(S + \sigma\kappa\right)$ with $\mathcal{L}_m K_{\alpha\beta}$

leads to

$$\begin{aligned}\left(\mathcal{L}_5 - \mathcal{L}_{\mathbf{N}}\right)K_{\mu\nu} &= -D_{\mu}D_{\nu}N \\ &\quad + N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_{\mu}^{\lambda}K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}\end{aligned}$$

# Decomposition of the Einstein Equations

## Constraint Equations

Projecting Einstein equations twice onto normal $n$

$$\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)n^\alpha n^\beta = \frac{8\pi G}{c^4}T_{\alpha\beta}n^\alpha n^\beta \longrightarrow R_{\alpha\beta}n^\alpha n^\beta - \frac{1}{2}\sigma R = \frac{8\pi G}{c^4}\kappa$$

Contracting indices in Gauss relation

$$R - 2\sigma R_{\alpha\beta}n^\alpha n^\beta = \bar{R} - \sigma\left(K^2 - K^{\alpha\delta}K_{\alpha\delta}\right)$$

Hamiltonian Constraint

Combining above leads to

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa$$

Momentum Constraint

Projecting Einstein equations onto $\sum_\tau$ and normal $n$

$$n^\alpha P^{\beta'}{}_{\beta}\left(R_{\alpha\beta'} - \frac{1}{2}g_{\alpha\beta'}R\right) = n^\alpha P^{\beta'}{}_{\beta}\frac{8\pi G}{c^4}T_{\alpha\beta'} = -\frac{8\pi G}{c^4}p_\beta$$

Combining with Codazzi relation leads to

$$D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4}p_\nu$$

# Decomposition of the Einstein Equations

## Summary of Differential Equations in 4+1 Formalism

Evolution equation for spacetime metric

$$\frac{1}{c_5}\mathcal{L}_\tau\,\gamma_{\mu\nu} - \mathcal{L}_\mathbf{N}\,\gamma_{\mu\nu} = -2NK_{\mu\nu}$$

Evolution equation for extrinsic curvature

$$\left(\frac{1}{c_5}\mathcal{L}_\tau - \mathcal{L}_\mathbf{N}\right)K_{\mu\nu} = -D_\mu D_\nu N + N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}$$

Hamiltonian Constraint

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa$$

Momentum Constraint

$$D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4}p_\nu$$

# Evolution Versus Constraints

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0$$

In $D$ dimensions symmetric tensor $G^{\alpha\beta}$ has $D(D+1)/2$ components

$G^{\alpha\beta}$ has 15 components on $\mathcal{M}_5$

5 components of Bianchi relation $\longrightarrow$ 10 independent components + 5 constraints

Order of $\tau$ derivative

Einstein equations $G_{\alpha\beta} = \dfrac{8\pi G}{c^4} T_{\alpha\beta} \ \longrightarrow$ 2$^{\text{nd}}$ order in $\tau$ derivatives of $g_{\alpha\beta}$

Initial conditions for evolution PDE $= \left\{ g_{\alpha\beta}\,,\, \partial_\tau g_{\alpha\beta}\,,\, T_{\alpha\beta} \right\}$

Bianchi relation

$$\nabla_\alpha G^{\alpha\beta} = 0 \ \longrightarrow\ \frac{1}{c_5} \partial_\tau G^{5\beta} = -\nabla_\mu G^{\mu\beta} + \{\text{Christoffel symbols}\} \times G^{\alpha\beta}$$

RHS at most 2$^{\text{nd}}$ order in $\partial_\tau \ \Longrightarrow\ G^{5\beta}$ at most 1$^{\text{st}}$ order in $\partial_\tau$

$G^{5\beta} \ \longrightarrow$ 5 constraints on initial conditions

Bianchi relation: $G^{\alpha\beta}\Big|_{\tau_0} = 0 \ \Rightarrow\ \dfrac{1}{c_5} \partial_\tau\, G^{5\beta}\Big|_{\tau_0} = 0 \ \longrightarrow$ constraint is conserved

# Static Schwarzschild-like Geometry

Metric for $T_{\alpha\beta} = 0 \longrightarrow S_{\mu\nu} = p_\mu = \kappa = 0$

$$ds^2 = -c^2 B dt^2 + A dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \sigma W c_5^2 d\tau^2$$

where

$$B(r) = A^{-1}(r) = 1 - \Phi_0(r) = 1 - \frac{GM}{rc^2} \qquad W = W(x,\tau)$$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} = \begin{bmatrix} \gamma_{\mu\nu}(r) & 0 \\ 0 & \sigma N^2(x,\tau) \end{bmatrix}$$

with $\bar{R} = 0$ for Schwarzschild spacetime metric and $N = \sqrt{W}$

Dynamical Equations $\longrightarrow$ 4D wave equation for $\sqrt{W}$

$$(\partial_5 - \mathcal{L}_{\mathbf{N}})\gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow \partial_\tau \gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow K_{\mu\nu} = 0$$

$$\partial_5 K_{\mu\nu} = -D_\mu D_\nu N + N\left(-\sigma \bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K_\mu^\lambda K_{\nu\lambda}\right) \longrightarrow \gamma^{\mu\nu} D_\mu D_\nu \sqrt{W(x,\tau)} = 0$$

Constraints trivially satisfied for $\bar{R} = K_{\mu\nu} = p_\nu = \kappa = 0$

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma \frac{16\pi G}{c^4}\kappa \qquad D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu$$

# Perturbation Around Schwarzschild Geometry

Admit variable mass

$$M(\tau) = M[1+\alpha(\tau)] \qquad \alpha^2 \ll 1 \longrightarrow B = A^{-1} = 1 - \Phi_0 [1+\alpha(\tau)]$$

Take $W = 1$

4D connection $\tau$-dependent but retains unperturbed form $\longrightarrow \bar{R} = 0$

Dynamical equations (neglecting terms in $\alpha^2$ and $\Phi_0^2$)

Metric evolution

$$\partial_5 \gamma_{\mu\nu} = -2NK_{\mu\nu} \longrightarrow K_{\mu\nu} = -\frac{1}{2c_5}\partial_\tau \gamma_{\mu\nu} = -\frac{\Phi_0 \dot{\alpha}(\tau)}{2c_5} \text{diag}\left(1, \frac{1}{B^2}, 0, 0\right)$$

$$K^\mu_\nu = \gamma^{\mu\lambda} K_{\lambda\nu} = -\frac{\Phi_0 \dot{\alpha}(\tau)}{2c_5} \text{diag}(-1,1,0,0) \longrightarrow K = K^\mu_\mu = 0$$

Curvature evolution, using $\bar{R} = 0$, $N = 1$, $N^\mu = 0$, and $(K_{\mu\nu})^2 \propto \alpha^2 \approx 0$

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2c_5^2}\Phi_0 \ddot{\alpha}(\tau) \text{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = \sigma \frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}(S + \sigma\kappa)\right]$$

# Perturbing Energy Momentum Tensor

Hamiltonian constraint $\longrightarrow$ no perturbing mass density

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa \ \longrightarrow\ \kappa \propto \alpha^2\Phi_0^2 \approx 0$$

Momentum constraint $\longrightarrow$ perturbing mass current into $r$ direction

$$p_\nu = D_\mu K^\mu_\nu - D_\nu K = \partial_\mu K^\mu_\nu + K^\lambda_\nu \Gamma^\mu_{\lambda\mu} - K^\mu_\lambda \Gamma^\lambda_{\nu\mu}$$

$$p_0 = \partial_r K^1_0 + K^0_0\Gamma^\mu_{0\mu} - K^\mu_\lambda\Gamma^\lambda_{0\mu} = K^0_0\Gamma^\mu_{0\mu} - K^0_0\Gamma^0_{00} - K^1_1\Gamma^1_{01} = 0$$

$$p_1 = \partial_r K^1_1 + K^\lambda_1\Gamma^\mu_{\lambda\mu} - K^\mu_\lambda\Gamma^\lambda_{1\mu} = -\frac{1}{2}\frac{1}{c_5 r}\Phi_0\dot{\alpha}(\tau)$$

Evolution equation $\longrightarrow$ perturbing energy density and momentum density in $r$ direction

$$\ddot{\alpha}(\tau)\ \ \Phi_0\,\mathrm{diag}\left(1, \frac{1}{B^2}, 0, 0\right) = -\sigma\frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}S\right]$$

$$S_{00} = S_{11} = \left(-\sigma\frac{c_5^2}{c^2}\frac{16\pi G}{c^2}\right)^{-1}\Phi_0\,\ddot{\alpha}(\tau) \qquad\qquad S_{22} = S_{33} = S = 0$$

# *Thank You For Your Patience*

Slides and preprints: http://cs.hac.ac.il/staff/martin