A Vielbein Formalism for SHP General Relativity

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Stueckelberg-Horwitz-Piron (SHP) Formalism

Covariant many-body canonical mechanics with parameterized evolution

Evolving 8D unconstrained phase space $\Longrightarrow$ scalar $\tau \neq$ proper time

$$x^\mu(\tau),\ \dot{x}^\mu(\tau) \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \qquad \lambda, \mu, \nu, \ldots = 0,1,2,3$$

Manifestly scalar Lagrangian

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - V(x) \qquad \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

Canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M g_{\mu\nu} \dot{x}^\nu \qquad \longrightarrow \qquad \dot{x}^\mu = \frac{1}{M} g^{\mu\nu} p_\nu$$

Manifestly scalar Hamiltonian

$$K = \dot{x}^\mu p_\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu + V(x) \qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x^\mu}$$

Stueckelberg-Horwitz-Piron (SHP) Formalism

Free Particle in Curved Spacetime

Lagrangian

$$L = \frac{1}{2} M g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \qquad \lambda, \mu, \nu = 0, 1, 2, 3$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\lambda}{D\tau} = \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu} \right)$$

Hamiltonian $K =$ total mass — unconstrained but conserved on geodesic

$$K = \dot{x}^\mu p_\mu - L = \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu \qquad \frac{dK}{d\tau} = M g_{\mu\nu} \dot{x}^\mu \frac{D\dot{x}^\nu}{D\tau} = 0$$

Poisson bracket — total mass conserved for $\partial_\tau g_{\mu\nu} = 0$

$$\frac{dK}{d\tau} = \{K, K\} + \frac{\partial K}{\partial \tau} = \frac{1}{2M} p_\mu p_\nu \ \frac{\partial g^{\mu\nu}}{\partial \tau} = 0$$

Local Dynamical Metric $g_{\mu\nu}(x,\tau)$

General relativity standing on one foot (J. A. Wheeler):

Spacetime tells matter how to move; matter tells spacetime how to curve.

Particle in SHP formalism: worldline traced out by event evolving with τ

Unconstrained trajectory $x^\mu(\tau),\ \dot{x}^\mu(\tau)$

Particle density $\rho(x,\tau)$

Energy-momentum tensor $T^{\mu\nu}(x,\tau)$

4D block universe $\mathcal{M}(\tau)$

Scalar Hamiltonian K generates evolution $\mathcal{M}(\tau) \longrightarrow \mathcal{M}(\tau+d\tau)$

Matter evolving in $\tau \longrightarrow$ curvature evolving in τ

Generalize Einstein equations for $g_{\mu\nu}(x,\tau)$

Approach to field equations — study classical SHP electrodynamics

SHP Offshell Electrodynamics

Make classical action locally gauge invariant

$$S_{\text{free}} \longrightarrow S_{\text{SHP}} = \int d\tau \; \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} \dot{x}^5 a_5(x,\tau)$$
$$= \int d\tau \; \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\alpha a_\alpha(x,\tau)$$

$\alpha, \beta, \gamma = 0,1,2,3,5$ $\quad$ $x^5 = c_5 \tau$ in analogy to $x^0 = ct$ $\quad$ $\dot{x}^5 =$ constant

S_{SHP} invariances

5D gauge invariance $\quad a_\alpha(x,\tau) \longrightarrow a_\alpha(x,\tau) + \partial_\alpha \Lambda(x,\tau)$

4D Lorentz invariance $\quad \dot{x}^\mu a_\mu$ and a_5 are O(3,1) scalars

Regard S_{SHP} as standard 5D action with symmetry breaking in matter term

$$S_{5\text{D}} = \int d\tau \; \frac{1}{2} M \dot{x}^\alpha \dot{x}_\alpha + \frac{e}{c} \dot{x}^\alpha a_\alpha \; \xrightarrow[\dot{x}^5 \,\equiv\, c_5]{} \; S_{\text{SHP}} = \int d\tau \; \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\alpha a_\alpha$$

Pose 5D Pseudo-spacetime $\mathcal{M}_5$

Particle dynamics

Free particle Lagrangian $L = \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{x}^\alpha \dot{x}^\beta$

Unbroken (incorrect) 5D equations of motion

$$\ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \qquad \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\beta\alpha} \right)$$

Break 5D symmetry to O(3,1): require $\dot{x}^5 = c_5$ non-dynamical (constant)

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \qquad \ddot{x}^5 \equiv 0$$

Field dynamics

5D gauge invariance of Ricci tensor $R_{\alpha\beta}$ $\longrightarrow$ 5D Bianchi identity

$$\text{Translation } x'^\alpha = x^\alpha + \Lambda^\alpha(x,\tau) \longrightarrow \nabla_\alpha \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0$$

Construct description of matter satisfying $\nabla_\beta T^{\alpha\beta} = 0 \longrightarrow$ field equations

Break 5D symmetry when equating field terms $R_{\alpha\beta}$ and matter terms $T^{\alpha\beta}$

Matter in $\mathcal{M}_5$

Non-interacting dust (pressure $= 0$)

Particle mass density $= \rho(x,\tau)$

5-component event current $= j^\alpha(x,\tau) = \dot{x}^\alpha(\tau)\rho(x,\tau)$

By geodesic equations

$$\nabla_\alpha j^\alpha = 0 \qquad\qquad \nabla_\alpha X^\beta = \partial_\alpha X^\beta + X^\gamma \Gamma^\beta_{\gamma\alpha}$$

Mass-energy-momentum tensor

$$\nabla_\beta T^{\alpha\beta} = 0 \qquad\qquad T^{\alpha\beta} = \rho \dot{x}^\alpha \dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = \rho \dot{x}^\mu \dot{x}^\nu \\ T^{5\beta} = c_5 j^\beta \end{cases}$$

Unbroken 5D Einstein equations and trace-reversed form

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta} \quad \longrightarrow \quad R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} + \frac{\frac{1}{2} g_{\alpha\beta} g^{\gamma'\delta'}}{1 - \frac{1}{2} g^{\gamma\delta} g_{\gamma\delta}} T_{\gamma'\delta'} \right)$$

Break 5D symmetry to O(3,1) at geometry/matter interface

Replace $g_{\gamma\delta} \longrightarrow \widehat{g}_{\gamma\delta}$ such that $\widehat{g}^{\gamma\delta} \widehat{g}_{\gamma\delta} = 4$

Structure of $\mathcal{M}_5$

Events $x_1 \in \mathcal{M}(\tau_1)$ and $x_2 \in \mathcal{M}(\tau_2)$

5D interval $dX = X_1 - X_2 = (x_1, c_5\tau_1) - (x_2, c_5\tau_2)$

Combines

Geometrical distance within $\mathcal{M}(\tau)$

Dynamical distance between $\mathcal{M}(\tau_1) \longrightarrow \mathcal{M}(\tau_2)$

Construct $\mathcal{M}_5$ as image of injective mapping

$$\Phi : \mathcal{M} \longrightarrow \mathcal{M}_5 = \mathcal{M} \times R \qquad\qquad X = \Phi(x,\tau) = (x, c_5\tau)$$

Characterize 4D spacetime $\mathcal{M}$ as hypersurface embedded in $\mathcal{M}_5$

Borrow mathematical tools of 3+1 Arnowitt Deser Misner (ADM) formalism

$$\left.\begin{array}{l}\text{Foliation of } \mathcal{M}_5 \text{ to hypersurfaces } \mathcal{M}(\tau) \\ \text{Vielbein frame for tangent space } \mathcal{T}(\mathcal{M}_5)\end{array}\right\} \xrightarrow[\text{systematic}]{} \widehat{g}_{\gamma\delta}$$

Quick Overview of ADM (With Apologies to Theory of Embedded Surfaces)

3+1 decomposition of 4D spacetime

Time evolution vector n_μ normal to 3D spacelike hypersurface Σ

Projection operator $P_{\mu\nu}$: 4D spacetime $\mathcal{M} \longrightarrow$ 3D space Σ

Induced 3D space metric on Σ: $\gamma_{ij} = P_{ij}$

Decomposition of geometrical objects and field equations

Projected covariant derivative D_μ and projected curvature $\bar{R}_{\mu\nu\lambda\rho}$ on Σ

Extrinsic curvature $K_{\mu\nu} =$ projected gradient of time vector n_μ

4D curvature $R_{\mu\nu\lambda\rho} \longrightarrow$ combinations of $\bar{R}_{\mu\nu\lambda\rho}$ and $K_{\mu\nu}$

Decompose $T_{\mu\nu} \longrightarrow T_{ij}$, momentum p_i, energy density E

Canonical formulation of GR — initial value problem for space metric

Lie derivatives along n_μ $\longrightarrow$ PDEs for γ_{ij} and K_{ij} first order in ∂_t

Hamiltonian from canonical conjugates γ_{ij} and $\pi_{ij} = \sqrt{\gamma}\left(K_{ij} - \gamma_{ij}K\right)$

Quintrad Frame for 5D Tangent Space

Coordinate frame

For tangent space $\mathcal{T}(\mathcal{M}_5)$ basis vectors $\mathbf{g}_\alpha = \partial_\alpha$

For cotangent space $\mathcal{T}^*(\mathcal{M}_5)$ basis 1-forms $\mathbf{g}^\alpha = \mathbf{d}X^\alpha$

$$\mathbf{g}^\alpha(\mathbf{g}_\beta) = \mathbf{g}^\alpha \cdot \mathbf{g}_\beta = \delta^\alpha_\beta \qquad \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta} \qquad \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = g^{\alpha\beta}$$

Quintrad frame

Constant vectors $\mathbf{e}_a$ for $\mathcal{T}(\mathcal{M}_5)$ and 1-forms $\mathbf{e}^a$ for $\mathcal{T}^*(\mathcal{M}_5)$

$$\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab} \qquad \mathbf{e}^a \cdot \mathbf{e}^b = \eta^{ab} \qquad \partial_a \mathbf{e}_b = \partial_a \mathbf{e}^b = 0$$

Latin indices $a, b, c, \cdots = 0, 1, 2, 3, 5$ indicate reference to quintrad

Vielbein field relates quintrad to position-dependent coordinate frame

$$\mathbf{g}_\alpha = E_\alpha{}^a \mathbf{e}_a \qquad \mathbf{e}_a = e^\alpha{}_a \mathbf{g}_\alpha \qquad e^\alpha{}_a E_\beta{}^a = \delta^\alpha_\beta$$

$$\mathbf{g}^\alpha = e^\alpha{}_a \mathbf{e}^a \qquad \mathbf{e}^a = E_\alpha{}^a \mathbf{g}^\alpha \qquad e^\alpha{}_a E_\alpha{}^b = \delta^b_a$$

Geometry in Quintrad

Transforming components

$$\mathbf{V} = V^\alpha \mathbf{g}_\alpha = \left(V^\alpha E_\alpha{}^a\right) \mathbf{e}_a = V^a \mathbf{e}_a \;\longrightarrow\; \begin{cases} V_b^a = E_\alpha{}^a e_a{}^\beta V_\beta^\alpha \\ V_\beta^\alpha = e^\alpha{}_a E_\beta{}^b V_b^a \end{cases}$$

Induced Metric

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \eta_{ab}\; E_\alpha{}^a E_\beta{}^b \qquad g^{\alpha\beta} = \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = \eta^{ab}\; e^\alpha{}_a e^\beta{}_b$$

$$\eta_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b = g_{\alpha\beta} e^\alpha{}_a e^\beta{}_b \qquad \eta^{ab} = \mathbf{e}^a \cdot \mathbf{e}^b = g^{\alpha\beta} E_\alpha{}^a E_\beta{}^b$$

Covariant derivative and curvature

$$\nabla_\alpha V_b^a = \partial_\alpha V_b^a + \omega_\alpha{}^a{}_c V_b^c - \omega_\alpha{}^c{}_b V_c^a$$

Spin connection $\omega_\alpha{}^b{}_a = -e^\beta{}_a \left(\partial_\alpha E_\beta{}^b\right) + E_\beta{}^b e^\gamma{}_a \Gamma^\beta_{\alpha\gamma}$

Compatibility $\nabla_\alpha E_\delta{}^b = 0$

Curvature $\left[\nabla_b, \nabla_a\right] V_c = V_d\, R^d_{cab}$

$$\boxed{\begin{aligned} R^d_{cab} = {} & \partial_a \omega_b{}^d{}_c - \partial_b \omega_a{}^d{}_c \\ & + \omega_a{}^d{}_{c'} \omega_b{}^{c'}{}_c - \omega_b{}^d{}_{c'} \omega_a{}^{c'}{}_c \end{aligned}}$$

Foliation
4+1 decomposition in coordinate frame

Scalar field $S(X) = X^5/c_5 = \tau$

Natural foliation defined by level surfaces $\Sigma_{\tau_0} = \{X^\alpha \mid S(X) = \tau_0\}$

Unit normal to Σ_{τ_0}

$$n_\alpha = \sigma \frac{1}{\sqrt{|g^{55}|}}\, \partial_\alpha S(X) = \sigma \frac{1}{\sqrt{|g^{55}|}} \delta^5_\alpha \qquad\qquad g^{\alpha\beta} n_\alpha n_\beta = \sigma = \pm 1$$

Coordinate frame for tangent space $\mathcal{T}(\Sigma_{\tau_0})$

$$(\mathbf{g}_\mu)^\alpha = \partial_\mu \Phi^\alpha = \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)_{\tau_0} = \delta^\alpha_\mu \quad \text{for } \mathcal{T}(\Sigma_{\tau_0}) \subset \mathcal{T}(\mathcal{M}_5)$$

Fifth basis vector for $\mathcal{T}(\mathcal{M}_5)$

Linear combination $\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + Nn$ (ADM parameterization)

$N^\mu =$ shift and $N =$ lapse (Lagrange multipliers)

Designate $\gamma_{\mu\nu} = g_{\mu\nu} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu$

Evaluate

$$g_{5\mu} = \mathbf{g}_\mu \cdot \mathbf{g}_5 = \mathbf{g}_\mu \cdot \left(N^{\mu'} \mathbf{g}_{\mu'} + Nn \right) = N_\mu$$

$$g_{55} = \mathbf{g}_5 \cdot \mathbf{g}_5 = \left(N^\mu \mathbf{g}_\mu + Nn \right) \cdot \left(N^{\mu'} \mathbf{g}_{\mu'} + Nn \right) = \gamma_{\mu\mu'} N^\mu N^{\mu'} + \sigma N^2$$

4+1 metric decomposition (ADM metric)

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}$$

$$g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix}$$

Foliation

4+1 decomposition in quintrad frame

Partition quintrad indices: $a, b, c, d, = 0, 1, 2, 3, 5 \qquad k, l, m, n, \ldots = 0, 1, 2, 3$

5-index in quintrad frame denoted $\bar{5}$

Quintrad: spacetime hypersurface spanned by $\{\mathbf{e}_k\}$ and normal to $\mathbf{e}_5$

Assign $\quad n = \mathbf{e}_5 \qquad \bar{n} = \sigma \mathbf{e}^5 \qquad n^2 = \eta_{55} = \bar{n}^2 = \eta^{55} = \sigma$

Frame transformations depend on $E_\mu{}^k$, N^μ, and N

$$\mathbf{g}_\alpha = E_\alpha{}^a \mathbf{e}_a = \delta^\mu_\alpha E_\mu{}^k \mathbf{e}_k + \delta^5_\alpha \left(E_\mu{}^k N^\mu \mathbf{e}_k + N \mathbf{e}_5 \right)$$

$$\mathbf{e}_a = e^\alpha{}_a \mathbf{g}_\alpha = \delta^k_a E^\mu{}_k \mathbf{g}_\mu + \delta^5_a \frac{1}{N} \left(-N^\mu \mathbf{g}_\mu + \mathbf{g}_5 \right)$$

Vielbein field

$$E_\alpha{}^a = \delta^\mu_\alpha \delta^a_k E_\mu{}^k + \delta^5_\alpha \left(E_\mu{}^k N^\mu \delta^a_k + N \delta^a_5 \right)$$

$$e^\alpha{}_a = \delta^k_a \delta^\alpha_\mu e^\mu{}_k - \delta^5_a \delta^\alpha_\mu \frac{1}{N} N^\mu + \delta^5_a \delta^\alpha_5 \frac{1}{N}$$

Dynamics in 4D spacetime vierbein field $E_\mu{}^k$

Projection Onto Hypersurface

Projection operator P acts on vector as

$$V \in \mathcal{T}\left(\mathcal{M}_5\right) \longrightarrow V_\perp = P\left[V\right] \in \mathcal{T}\left(\Sigma_\tau\right) \subset \mathcal{T}\left(\mathcal{M}_5\right)$$

$$P_{aa'} = \eta_{aa'} - \sigma n_a n_{a'} = \eta_{aa'} - \sigma \delta_a^5 \delta_{a'}^5 \qquad P_{a'}^a = \delta_{a'}^a - \delta_5^a \delta_{a'}^5$$

Pull-back and push-forward

$$v^\mu = \delta_\alpha^\mu e^\alpha{}_a V_\perp^a \in \mathcal{T}\left(\mathcal{M}\right) = \delta_\alpha^\mu e^\alpha{}_{a'} P_a^{a'} V^a \qquad V_\perp^a = \delta_\mu^\alpha E_\alpha{}^a v^\mu \in \mathcal{T}\left(\Sigma_\tau\right)$$

Completeness relations

$$\eta_{aa'} = P_{aa'} + \sigma \delta_a^5 \delta_{a'}^5 \qquad \delta_{a'}^a = P_{a'}^a + \delta_5^a \delta_{a'}^5$$

On Σ_τ hypersurface

$$g_{\alpha\beta} = P_{\alpha\beta} + \sigma n_\alpha n_\beta \ \longrightarrow \ \text{pull-back} \ \ \gamma_{\mu\nu} = \delta_\mu^\alpha \delta_\nu^\beta \, g_{\alpha\beta} = P_{\mu\nu}$$

Simplification: compute using $P_{\alpha\beta}$ and pull back to Σ_τ

SHP Field Equations
Quintrad frame

Decompose the mass-energy-momentum tensor

$$T_{ab} = T_{a'b'}\left(P_a^{a'} + \sigma n^{a'} n_a\right)\left(P_b^{b'} + \sigma n^{b'} n_b\right) = S_{ab} - 2\sigma n_a p_b + n_a n_b \kappa$$

$$S_{ab} = T_{a'b'} P_a^{a'} P_b^{b'} \qquad p_b = -n^{a'} P_b^{b'} T'_{a'b'} \qquad \kappa = n^{b'} n^{a'} T_{a'b'}$$

Trace-reversed form of 5D field equations

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} + \frac{\frac{1}{2}\eta_{ab}}{1 - \frac{1}{2}\eta^{cd}\eta_{cd}}\eta^{c'd'} T_{c'd'}\right) \qquad \eta^{ab}\eta_{ab} = 5$$

Break 5D symmetry: $\eta_{ab} \longrightarrow \widehat{\eta}_{ab} = \delta_a^k \delta_b^l \eta_{kl} \longrightarrow \widehat{\eta}^{ab}\widehat{\eta}_{ab} = 4$

SHP field equations

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \frac{1}{2}\widehat{\eta}_{ab}\widehat{T}\right) \qquad \widehat{T} = \widehat{\eta}^{ab} T_{ab} = \eta^{kl} T_{kl} = \eta^{ab} S_{ab} = S$$

SHP Field Equations
Coordinate frame

Transform unbroken η_{ab} to unbroken local metric

$$g_{\alpha\beta} = E_\alpha{}^a E_\beta{}^b \eta_{ab} = \delta_\alpha^\mu \delta_\beta^{\mu'} \gamma_{\mu\mu'} + \delta_\alpha^5 \delta_\beta^\mu N_\mu + \delta_\alpha^\mu \delta_\beta^5 N_\mu + \delta_\alpha^5 \delta_\beta^5 \left(N^\mu N_\mu + \sigma N^2 \right)$$

Recovers 5D ADM metric

Transform broken $\widehat{\eta}_{ab}$ to broken local metric

$$\widehat{g}_{\alpha\beta} = E_\alpha{}^a E_\beta{}^b \widehat{\eta}_{ab} = g_{\alpha\beta} - \delta_\alpha^5 \delta_\beta^5 \sigma N^2 = g_{\alpha\beta} - \sigma n_\alpha n_\beta = P_{\alpha\beta}$$

SHP field equations

$$R_{\alpha\beta} = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} S \right)$$

$$\boxed{R_{\alpha\beta} - \frac{1}{2} P_{\alpha\beta} \widehat{R} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$

$$\widehat{R} = P^{\alpha\beta} R_{\alpha\beta}$$

Preserves 5D gauge symmetry

O(3,1) covariant with scalar 5-index

Decomposition of Curvature

Projected covariant derivative

$$(DX)_{ab_1\cdots b_n} = P_a^{a'} P_{b_1}^{b'_1} \ \cdots \ P_{b_n}^{b'_n} \left(\nabla_{a'} X_{b'_1 \cdots b'_n} \right)$$

Compatible with metric $\;(DP)_{abc} = P_a^{a'} P_b^{b'} P_c^{c'} \ \nabla_{a'} P_{b'c'} = 0$

Extrinsic curvature

$$K_{ab} = -\left(Dn\right)_{ab} = -P_a^{a'} P_b^{b'} \nabla_{b'} n_{a'} = -P_b^{b'} \nabla_{b'} n_a = \sigma \delta_a^k \delta_b^l \ \omega_l{}^{\bar{5}}{}_k$$

Projected curvature tensor

Expand $[D_b, D_a]\, X_c = X_d \bar{R}^d_{cab}$

Gauss relation $\;\left(P^{a'}{}_a P^{b'}{}_b P^c{}_{c'} P^{d'}{}_d \right) R^{c'}{}_{d'a'b'} = \bar{R}^c_{dab} - \sigma \left(K_a^c K_{bd} - K_b^c K_{ad} \right)$

Codazzi relation $\;\left(P_a^{a'} P_b^{b'} P_c^{c'} \right) n_\delta \ R^\delta_{c'a'b'} = D_b K_{ac} - D_a K_{bc}$

$$\left(P_{aa'} P_b^{b'} \right) n^{c'} n^d \ R^{a'}_{db'c'} = -K_a^c K_{cb} - \sigma \frac{1}{N} D_b D_a N + P_a^{a'} P_b^{b'} n^{c'} \nabla_{c'} K_{a'b'}$$

Initial Value Problem in Coordinate Frame

Lie derivative

$$\text{Along} \quad m = Nn = \mathbf{g}_5 - \mathbf{N} \quad \longrightarrow \quad \mathcal{L}_m = \mathcal{L}_{\mathbf{g}_5} - \mathcal{L}_{\mathbf{N}}$$

$$\mathcal{L}_{\mathbf{g}_5} A_{\alpha\beta} = \delta_5^\gamma \partial_\gamma A_{\alpha\beta} + A_{\gamma\beta} \partial_\alpha \delta_5^\gamma + A_{\alpha\gamma} \partial_\beta \delta_5^\gamma = \partial_5 A_{\alpha\beta} = \frac{1}{c_5} \partial_\tau A_{\alpha\beta}$$

Evaluate $\mathcal{L}_m \, \widehat{g}_{\alpha\beta} = \mathcal{L}_m \, P_{\alpha\beta}$ and pull back to $\mathcal{M}$

$$\boxed{\frac{1}{c_5} \partial_\tau \gamma_{\mu\nu} = \mathcal{L}_{\mathbf{N}} \, \gamma_{\mu\nu} - 2N K_{\mu\nu}}$$

Evaluate $\mathcal{L}_m \, K_{\alpha\beta}$ using Gauss, 2-n projection and SHP field equations for $R_{\alpha\beta}$

$$\boxed{\begin{aligned} \frac{1}{c_5} \partial_\tau K_{\mu\nu} &= -D_\mu D_\nu N + \mathcal{L}_{\mathbf{N}} K_{\mu\nu} \\ &\qquad + N \left\{ -\sigma \bar{R}_{\mu\nu} + K K_{\mu\nu} - 2 K_\mu^\lambda K_{\nu\lambda} + \sigma \frac{8\pi G}{c^4} \left(S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} S \right) \right\} \end{aligned}}$$

Gauss + Codazzi relations $\longrightarrow$ Hamiltonian and momentum constraints

$$\boxed{\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -\frac{8\pi G}{c^4} \left(S + \sigma \kappa \right) \qquad D_\mu K_\nu^\mu - D_\nu K = \frac{8\pi G}{c^4} p_\nu}$$

Perturbation of flat metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad \longrightarrow \qquad \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \qquad \left(h_{\alpha\beta}\right)^2 \approx 0$$

Choose Lorenz gauge $\partial^\beta \left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h \right) = 0$

SHP field equation $\longrightarrow$ $\boxed{R_{\alpha\beta} \approx -\frac{1}{2}\partial^\gamma \partial_\gamma h_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\widehat{\eta}_{\alpha\beta} S \right)}$

Approximations for 4+1 decomposition

$$\left(K_{\mu\nu}\right)^2 \approx 0 \qquad \qquad N K_{\mu\nu} \approx 0 \qquad \qquad \mathcal{L}_{\mathbf{N}} K_{\mu\nu} \approx 0$$

$$\frac{1}{c_5}\partial_\tau \gamma_{\mu\nu} \approx \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - 2K_{\mu\nu} \;\longrightarrow\; 2K_{\mu\nu} = \underbrace{\partial_\mu h_{5\nu} + \partial_\nu h_{5\mu} - \partial_5 \gamma_{\mu\nu}}_{2\sigma\Gamma^5_{\mu\nu}}$$

Satisfied automatically: $K_{ab} = \omega_{l}{}^{\bar{5}}{}_{k} \longrightarrow K_{\mu\nu} = \sigma\Gamma^5_{\mu\nu}$

Weak Gravitation

Remaining 4+1 equations equivalent to wave equation

Evolution equation

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\sigma \bar{R}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)$$

Equivalent to spacetime part of SHP field equations

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S\right)$$

Bianchi identity $\longrightarrow R_{5\alpha}$ components are non-dynamical constraints

Hamiltonian constraint

$$\bar{R} \approx -\frac{8\pi G}{c^4}\left(S + \sigma\kappa\right) \quad \text{equivalent to} \quad R_{55} = \frac{8\pi G}{c^4}T_{55}$$

Momentum constraint

$$\partial_\mu K^\mu_\nu - \partial_\nu K \approx \frac{8\pi G}{c^4}p_\nu \quad \text{equivalent to} \quad R_{5\mu} = \frac{8\pi G}{c^4}T_{5\mu}$$

4D Metric $\gamma_{\mu\nu}(x,\tau)$ on $\mathcal{M}$

4+1 (ADM) decomposition of any 5D metric $g_{\alpha\beta}(x,\tau)$

$$g_{\alpha\beta} = \begin{bmatrix} g_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + g_{\mu\nu} N^\mu N^\nu \end{bmatrix}$$

Broken-symmetry SHP metric

$$\widehat{g}_{\alpha\beta} = P_{\alpha\beta} = g_{\alpha\beta} - \delta^5_\alpha \delta^5_\beta \sigma N^2 = \begin{bmatrix} g_{\mu\nu} & N_\mu \\ N_\mu & g_{\mu\nu} N^\mu N^\nu \end{bmatrix}$$

Pull back to $\mathcal{M}$

$$\gamma_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu \; \widehat{g}_{\alpha\beta} = g_{\mu\nu}(x,\tau)$$

Independent of non-dynamical lapse N and shift N_μ

Example — Field Induced By Event

Event evolving on t-axis as $X^\mu(\tau) = (ct, \mathbf{0}) = (c\zeta(\tau), \mathbf{0})$

Density $\rho(x,\tau) = \delta^{(3)}(\mathbf{x})\,\varphi\big(t - \zeta(\tau)\big)$

Mass-energy-momentum tensor $T^{\mu\nu} = \delta_0^\mu \delta_0^\nu\, mc^2 \rho(x,\tau)\, \dot{\zeta}^2(\tau)$

First-order solution to wave equation with $2Gm/c^2 r \ll 1$

$$\gamma_{\mu\nu} \approx \text{diag}\left(-B, \frac{1}{B}\delta_{ij}\right) \qquad\qquad B = 1 - \frac{2Gm}{c^2 r}\varphi\big(t - \zeta(\tau)\big)\dot{\zeta}^2(\tau)$$

Transform to spherical coordinates (x^0, r, θ, ϕ)

$$ds^2 = -B\,c^2 dt^2 + \frac{1}{B}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$

$\dot{\zeta} = \varphi = 1 \;\longrightarrow\; ds^2 \approx$ Schwarzschild in isotropic coordinates

$\gamma_{\mu\nu} = \tau$-perturbed Schwarzschild metric $\qquad m(t,\tau) = m\varphi\big(t - \zeta(\tau)\big)\dot{\zeta}^2(\tau)$

Full solution for $\gamma_{\mu\nu}(x,\tau)$ requires τ-evolution equations

Thank You For Your Patience

Slides and preprints: http://cs.hac.ac.il/staff/martin