# Speeds of Light and Mass Stability in Stueckelberg-Horwitz-Piron (SHP) Electrodynamics

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# Background

## Formal approach to special relativity

Define flat metric in Minkowski space

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \qquad g_{\mu\nu} = diag(-1,1,1,1)$$

Parameterize by proper time $\Rightarrow$ constraint on four-velocity

$$-1 = \frac{dx^\mu}{ds}\frac{dx_\mu}{ds} = -\left(\frac{dx^0}{ds}\right)^2 + \left(\frac{d\mathbf{x}}{ds}\right)^2 = -\left(\frac{dx^0}{ds}\right)^2 \left[1 - \left(\frac{d\mathbf{x}}{dx^0}\right)^2\right]$$

$$\frac{dx^0}{ds} = \gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}} = \frac{1}{\sqrt{1-\dfrac{\mathbf{v}^2}{c^2}}} \xrightarrow[\text{NR limit: } c\to\infty]{} 1$$

Mass-shell constraint

$$m^2 = -\left(m\frac{dx^\mu}{ds}\right)\left(m\frac{dx_\mu}{ds}\right) = -p^2 = E^2 - \mathbf{p}^2$$

# Background

## Covariant canonical mechanics — Stueckelberg (1941), Horwitz and Piron (1970)

8D unconstrained phase space $\left(x^\mu(\tau), \dot{x}^\mu(\tau)\right)$ $\qquad \dot{x}^\mu = dx^\mu / d\tau$

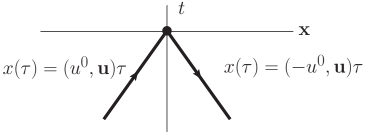
- New evolution parameter $\tau$
- Independent of spacetime coordinates (not proper time)

Distinguish two aspects of time:

- Chronological time $\tau$ determines monotonic ordering of events
- Coordinate time $x^0$ locates event on laboratory clock

$$E = M\dot{x}^0 = M\frac{dx^0}{d\tau}$$

Particle: $\dot{x}^0 > 0$ $\qquad$ Antiparticle: $\dot{x}^0 < 0$




Upgrade nonrelativistic classical and quantum mechanics

$$\left.\begin{array}{c}\text{Newtonian time } t \\ + \\ \text{Galilean symmetry}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Evolution parameter } \tau \\ + \\ \text{Poincaré symmetry}\end{array}\right.$$

## Dynamical U(1) gauge theory of spacetime events

- Event $x^\mu(\tau)$ evolves under chronological time $\tau$, $\mu,\nu = 0,1,2,3$
- $x^\mu(\tau) \longrightarrow$ a 5D current $j^\alpha(x,\tau)$, $\alpha,\beta,\gamma = 0,1,2,3,5$
- $j^\alpha(x,\tau) \longrightarrow$ 10 field strengths $f^{\alpha\beta}(x,\tau)$ through pre-Maxwell equations

$$\partial_\beta f^{\alpha\beta}(x,\tau) = ej^\alpha(x,\tau) \qquad \partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0$$

- Fields act on events through Lorentz force

$$M\ddot{x}^\mu(\tau) = e_0 f^\mu{}_\alpha(x,\tau)\dot{x}^\alpha(\tau) = e_0\left[f^\mu{}_\nu(x,\tau)\dot{x}^\nu(\tau) + f^\mu{}_5(x,\tau)\right]$$

- Particles and fields can exchange mass through

$$\frac{d}{d\tau}(-\tfrac{1}{2}M\dot{x}^2) = -M\dot{x}^\mu\ddot{x}_\mu = e_0 f_{5\mu}\dot{x}^\mu$$

- Particles and fields exchange mass dynamically.
  Why do observed particles have fixed masses?

- Field equations $\longrightarrow$ wave equation

$$g^{\alpha\beta}\partial_\alpha\partial_\beta a^\gamma = (\partial_\mu\partial^\mu + g^{55}\,\partial_\tau^2)a^\gamma = -ej^\gamma\,(x,\tau)$$

  $g^{\alpha\beta} = (-1,1,1,1,\pm1)$ suggests underlying symmetry of fields

$$\text{O(4,1) or O(3,2)} \xrightarrow[\text{in presence of matter}]{} \text{O(3,1)}$$

  How should $g_{55}$ be chosen?

- $c \to \infty$: standard relativity $\longrightarrow$ nonrelativistic mechanics.
  Is there a parameter $c_5$ such that

$$\text{SHP} \xrightarrow[c_5 \to \text{appropriate limit}]{} \text{standard Maxwell theory}$$

# Background

## Outline of Talk

- Overview of SHP electrodynamics making $c$ and $c_5$ explicit
- SHP $\xrightarrow[c_5 \to 0]{}$ standard Maxwell
- Simple model for particle mass shift $\longrightarrow$ off-shell evolution
- Self-interaction that damps out mass shift $\longrightarrow$ on-shell evolution

# Speeds of light in SHP

Constants $c$ and $c_5$ convert times $t$ and $\tau$ to spatial distance

## Generalized Stueckelberg-Schrodinger equation

$$\left(i\hbar\partial_\tau + \frac{e_0}{c}\phi\right)\ \psi\left(x,\tau\right) = \frac{1}{2M}\left(p^\mu - \frac{e_0}{c}a^\mu\right)\left(p_\mu - \frac{e_0}{c}a_\mu\right)\ \psi\left(x,\tau\right)$$

## Invariant under local gauge transformations

$$\psi(x,\tau) \to \exp\left[\frac{ie_0}{\hbar c}\Lambda(x,\tau)\right]\psi(x,\tau)$$

$$a_\mu(x,\tau) \to a_\mu(x,\tau) + \partial_\mu\Lambda(x,\tau)$$

$$\phi(x,\tau) \to \phi(x,\tau) + \partial_\tau\Lambda(x,\tau)$$

## Global gauge invariance $\longrightarrow$ conserved 5D current $\qquad \partial_\mu j^\mu + \partial_\tau\rho = 0$

$$j^\mu = -\frac{i\hbar}{2M}\left\{\psi^*\left(\partial^\mu - \frac{ie_0}{c}a^\mu\right)\psi - \psi\left(\partial^\mu + \frac{ie_0}{c}a^\mu\right)\psi^*\right\} \qquad \rho = \left|\psi(x,\tau)\right|^2$$

Formal designations:

$$\left.\begin{array}{l} x^5 = c_5\tau \quad , \quad \partial_5 = \dfrac{1}{c_5}\partial_\tau \\[2ex] j^5 = c_5\rho \quad , \quad a_5 = \dfrac{1}{c_5}\phi \end{array}\right\} \longrightarrow \left\{\begin{array}{l} a_\alpha \to a_\alpha + \partial_\alpha\Lambda \\[2ex] \partial_\alpha j^\alpha = 0 \end{array}\right.$$

## Hamiltonian

$$K = \left[\frac{1}{2M}\left(p^\mu - \frac{e_0}{c}a^\mu\right)\left(p_\mu - \frac{e_0}{c}a_\mu\right) - \frac{e_0}{c}\phi\right] \longrightarrow \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = \frac{1}{M}\left(p_\mu - \frac{e_0}{c}a_\mu\right)$$

## Lagrangian and Lorentz force

$$L = \dot{x}^\mu p_\mu - K = \frac{1}{2}M\dot{x}^\mu\dot{x}_\mu + \frac{e_0}{c}\dot{x}^\alpha a_\alpha$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0 \quad \longrightarrow \quad \frac{d}{d\tau}\left(M\,\dot{x}^\mu + \frac{e_0}{c}a^\mu\right) = \partial^\mu\left(\frac{e_0}{c}\dot{x}^a a_a\right)$$

$$M\ddot{x}^\mu = \frac{e_0}{c}\left[\dot{x}^\alpha\partial^\mu a_\alpha - \dot{x}^\alpha\partial_\alpha a^\mu\right] = \frac{e_0}{c}f^\mu{}_\alpha(x,\tau)\dot{x}^\alpha = \frac{e_0}{c}f^\mu{}_\nu(x,\tau)\dot{x}^\nu + \frac{e_0 c_5}{c}f^\mu{}_5(x,\tau)$$

where

$$f^\mu{}_\nu = \partial^\mu a_\nu - \partial_\nu a^\mu \qquad f^\mu{}_5 = \partial^\mu a_5 - \frac{1}{c_5}\partial_\tau a^\mu \qquad \dot{x}^5 = c_5\dot{\tau} = c_5$$

Unconstrained motion in 4D

$$\dot{x}^2 = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right)^2 = c^2\dot{t}^2\left(1, \frac{1}{c}\left(\frac{d\mathbf{x}}{d\tau}\right)\left(\frac{dt}{d\tau}\right)^{-1}\right)^2 = -c^2\dot{t}^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right)$$

On-shell evolution

$$\left|\frac{dt}{d\tau}\right| = |\dot{t}| = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \qquad \Rightarrow \qquad \dot{x}^2 = -c^2$$

Particles and fields may exchange mass $\longrightarrow$ off-shell evolution

$$\frac{d}{d\tau}(-\tfrac{1}{2}M\dot{x}^2) = -M\dot{x}^\mu\ddot{x}_\mu = -\frac{e_0}{c}\,\dot{x}^\mu(c_5 f_{\mu 5} + f_{\mu\nu}\dot{x}^\nu) = \frac{e_0 c_5}{c}\,\dot{x}^\mu f_{5\mu}$$
$$= g_{55}\, e_0 \frac{c_5}{c}\, f^{5\mu}\dot{x}_\mu$$

Mass exchange scaled by $\dfrac{c_5}{c}$

Add kinetic term for gauge field

$$S_{\rm em} = \int d^4x d\tau \left\{ \frac{e_0}{c} j^\alpha(x,\tau) a_\alpha(x,\tau) - \int ds\, \frac{\lambda}{4c} \left[ f^{\alpha\beta}(x,\tau) \Phi(\tau - s) f_{\alpha\beta}(x,s) \right] \right\}$$

Local event current

$$j^\alpha(x,\tau) = c\dot{X}^\alpha(\tau)\delta^4\left(x - X(\tau)\right)$$

Field interaction kernel

$$\Phi(\tau) = -\delta(\tau) - (\alpha\lambda)^2 \delta''(\tau) = \int \frac{d\kappa}{2\pi} \left[ 1 + (\alpha\lambda\kappa)^2 \right] e^{-i\kappa\tau} \,, \qquad \alpha = \frac{1}{2} \left[ 1 + \left( \frac{c_5}{c} \right)^2 \right]$$

Inverse kernel

$$\varphi(\tau) = \Phi^{-1}(\tau) = \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa\tau}}{1 + (\alpha\lambda\kappa)^2} = \frac{1}{2\alpha\lambda} e^{-|\tau|/\alpha\lambda} \,, \qquad \alpha\lambda \sim \text{width of } \varphi(\tau)$$

Smoothed Current

$$j^\alpha_\varphi(x,\tau) = \int ds\; \varphi(\tau - s)\, j^\alpha(x,s)$$

## 5D pre-Maxwell equations

$$\partial_\beta f^{\alpha\beta}(x,\tau) = \frac{e}{c}\, j_\varphi^\alpha(x,\tau) \qquad \partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0$$

## 4D component form

$$\partial_\nu\, f^{\mu\nu} - \frac{1}{c_5}\partial_\tau\, f^{5\mu} = \frac{e}{c}\, j_\varphi^\mu \qquad \partial_\mu\, f^{5\mu} = \frac{e}{c}\, j_\varphi^5 = \frac{c_5}{c}\, e\rho_\varphi$$

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \qquad \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5}\partial_\tau f_{\mu\nu} = 0$$

## Analog of 3-vector Maxwell equations

$$\nabla \times \mathbf{B} - \frac{1}{c}\partial_t \mathbf{E} = \frac{e}{c}\, \mathbf{J} \qquad \nabla \cdot \mathbf{E} = \frac{e}{c}\, J^0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{1}{c}\partial_t \mathbf{B} = 0$$

## 3-vector notation

$$(\mathbf{e})^i = f^{0i} \qquad (\mathbf{h})_i = \epsilon_{ijk} f^{jk} \qquad (\mathbf{f}^5)^i = f^{5i}$$

Equilibrium boundary conditions

$$\rho_\varphi(x,\tau) \xrightarrow[\tau\to\pm\infty]{} 0 \qquad f^{5\mu}(x,\tau) \xrightarrow[\tau\to\pm\infty]{} 0$$

Integration over worldline

$$\left.\begin{array}{c} \partial_\beta f^{\alpha\beta}(x,\tau) = \dfrac{e}{c} j^\alpha_\varphi(x,\tau) \\ \partial_{[\alpha} f_{\beta\gamma]} = 0 \\ \partial_\alpha j^\alpha = 0 \end{array}\right\} \xrightarrow[\int d\tau]{} \left\{\begin{array}{c} \partial_\nu F^{\mu\nu}(x) = \dfrac{e}{c} J^\mu(x) \\ \partial_{[\mu} F_{\nu\rho]} = 0 \\ \partial_\mu J^\mu(x) = 0 \end{array}\right.$$

where

$$A^\mu(x) = \int d\tau\; a^\mu(x,\tau) \qquad F^{\mu\nu}(x) = \int d\tau\; f^{\mu\nu}(x,\tau) \qquad J^\mu(x) = \int d\tau\; j^\mu(x,\tau)$$

Concatenation $\longrightarrow$ on-shell Maxwell theory $\sim$ equilibrium limit

$$\text{dim}\{e_0 a^\mu\} = \text{dim}\{e A^\mu\} \Rightarrow \begin{cases} \text{dim}\{e_0\} = \text{dim}\{\lambda\} = \text{time} \\ e = e_0/\lambda \text{ is dimensionless Maxwell charge} \end{cases}$$

## Wave equation

$$\partial_\beta \partial^\beta a^\alpha = \left(\partial_\mu \partial^\mu + \partial_\tau \partial^\tau\right) a^\alpha = \left(\partial_\mu \partial^\mu + \frac{g_{55}}{c_5^2}\,\partial_\tau^2\right) a^\alpha = -\frac{e}{c}\; j_\varphi^\alpha\,(x,\tau)$$

## Principal part Greens function

$$G_P(x,\tau) = -\frac{1}{4\pi}\delta(x^2)\delta(\tau) - \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-g_{55}g_{\alpha\beta}x^\alpha x^\beta)\frac{1}{\sqrt{-g_{55}g_{\alpha\beta}x^\alpha x^\beta}}$$

$$= \; G_{Maxwell} + G_{Correlation}$$

where under concatenation

$$\int d\tau\; G_{Maxwell} = D(x) = -\frac{1}{4\pi}\delta(x^2) \qquad\qquad \int d\tau\; G_{Correlation} = 0$$

## Support of $G_{Correlation}$

$$-g_{55}g_{\alpha\beta}x^\alpha x^\beta = \begin{cases} -\left(x^2 + c_5^2\tau^2\right) = c^2t^2 - \mathbf{x}^2 - c_5^2\tau^2 > 0 \quad , & g_{55} = +1 \\ +\left(x^2 - c_5^2\tau^2\right) = \mathbf{x}^2 - c^2t^2 - c_5^2\tau^2 > 0 \quad , & g_{55} = -1 \end{cases}$$

Taking $g_{55} = +1$ for timelike support $\longrightarrow$ finite self-interaction

# 'Static' Coulomb Potential

## Uniformly moving source

Source moving along $t$-axis

$$X(\tau) = (c\tau, 0, 0, 0)$$

Currents

$$j^0(x,\tau) = j^5(x,\tau) = c\delta(t-\tau)\,\delta^3(\mathbf{x}) \qquad \mathbf{j}(x,\tau) = 0$$

$$j^0_\varphi(x,\tau) = j^5_\varphi(x,\tau) = c\varphi(t-\tau)\,\delta^3(\mathbf{x}) \qquad \mathbf{j}_\varphi(x,\tau) = 0$$

Maxwell part of Green's function induces

$$a^0(x,\tau) = a^5(x,\tau) = \frac{e}{4\pi|\mathbf{x}|}\varphi\left(\tau - \left(t - \frac{|\mathbf{x}|}{c}\right)\right) \qquad \mathbf{a} = 0$$

Concatenation recovers standard Coulomb potential

$$A^0(x) = \int d\tau\, a^0(x,\tau) = \frac{e}{4\pi|\mathbf{x}|} \qquad \mathbf{A} = 0$$

Yukawa-type potential with photon mass spectrum $m_\gamma \sim \hbar/\alpha\lambda c^2$

$$a^0(x,\tau) = a^5(x,\tau) = \frac{e}{4\pi|\mathbf{x}|}\frac{1}{2\alpha\lambda}e^{-|\mathbf{x}|/\alpha\lambda c}$$

# 'Static' Coulomb Potential

## Coulomb force

Field strength components

$$f^{k0}(x,\tau) = f^{k5}(x,\tau) = \partial^k \frac{e}{4\pi|\mathbf{x}|} \frac{1}{2\alpha\lambda} e^{-|\mathbf{x}|/\alpha\lambda c} \qquad f^{ij}(x,\tau) = 0 \qquad f^{50} = 0$$

Test event experiences Coulomb force

$$M\ddot{x}^k = \frac{e_0}{c} f^k{}_\nu \dot{x}^\nu - g_{55} \frac{e_0 c_5}{c} f^{5k} = -\frac{e_0}{c} f^{k0} \left(\dot{x}^0 - g_{55} c_5\right)$$

Writing $\dot{x}(\tau) = (\pm c, \mathbf{0})$ for a particle $(+)$ or antiparticle $(-)$

$$M\ddot{\mathbf{x}} = -\frac{e_0 e}{2\alpha\lambda} \left(\pm 1 - g_{55} \frac{c_5}{c}\right) \nabla \left( \frac{e^{-|\mathbf{x}|/\alpha\lambda c}}{4\pi\,|\mathbf{x}|} \right) = \mp e^2 \, \frac{1 \mp g_{55} \frac{c_5}{c}}{1 + \left(\frac{c_5}{c}\right)^2} \nabla \left( \frac{e^{-|\mathbf{x}|/\alpha\lambda c}}{4\pi\,|\mathbf{x}|} \right)$$

If $c_5 = c$ then $\begin{cases} g_{55} = +1 \;\Rightarrow\; \text{no low energy particle-particle interaction} \\ g_{55} = -1 \;\Rightarrow\; \text{no low energy particle-antiparticle interaction} \end{cases}$

$\frac{c_5}{c} <$ measured charge asymmetry in low energy scattering

# 'Static' Coulomb Potential

## Contribution from $G_{correlation}$

Approximating $\varphi(\tau' - s) = \delta(\tau' - s)$

$$a^{\alpha}(x,\tau) = \frac{e}{2\pi^2}\ \dot{X}^{\alpha}(s)\int ds\ G_{correlation}\left(x - X(s), \tau - s\right)\theta\left(x^0 - X^0(s)\right)$$

Taking $g_{55} = 1$

$$G_{correlation}(x,\tau) = -\frac{c_5}{2\pi^2}\left(\frac{1}{2}\frac{\theta(-x^2 - c_5^2\tau^2)}{\left(-x^2 - c_5^2\tau^2\right)^{3/2}} - \frac{\delta\left(-x^2 - c_5^2\tau^2\right)}{\left(-x^2 - c_5^2\tau^2\right)^{1/2}}\right)$$

Solving causality requirements on $s$

$$a(x,\tau) = \frac{ec_5}{2\pi^2 c^3}\left(1, \mathbf{0}, \frac{c_5}{c}\right)\left(\frac{1}{2}\int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} - \int_{-\infty}^{\infty} ds \frac{\delta\left(g(s)\right)}{g^{1/2}(s)}\theta(t - s)\right)$$

where

$$s_- = \frac{\left(t - \left(\frac{c_5}{c}\right)^2\tau\right) - \sqrt{\frac{R^2}{c^2}\left(1 - \left(\frac{c_5}{c}\right)^2\right) + \left(\frac{c_5}{c}\right)^2 (t-\tau)^2}}{\left(1 - \left(\frac{c_5}{c}\right)^2\right)}, \quad g(s) = (t - s)^2 - \frac{R^2}{c^2} - \left(\frac{c_5}{c}\right)^2(\tau - s)^2$$

and so

$$a_{correlation}(x,\tau) = \frac{e}{4\pi^2}\left(c, \mathbf{0}, c_5\right)\frac{c_5}{c}\frac{\sqrt{1 - \frac{c_5}{c}}}{R^2\left(1 - \frac{c_5}{c}\right) + \frac{c_5}{c}c^2(t-\tau)^2} \sim \frac{c_5}{c}\frac{1}{R^2}$$

Integrating $\delta$-functions in $G_{Maxwell}(x,\tau)$ and $j^{\alpha}_{\varphi}(x,\tau)$ with

$$\int d\tau f(\tau)\,\delta[g(\tau)] = \frac{f(\tau_R)}{|g'(\tau_R)|}\ , \qquad g(\tau_R) = 0$$

leads to

$$a^{\alpha}(x,\tau) = \frac{e}{2\pi}\int ds\ \varphi(\tau - s)\,\dot{X}^{\alpha}(s)\,\delta\left((x - X(s))^2\right)\theta^{ret}$$
$$= \frac{e}{4\pi}\varphi(\tau - \tau_R)\,\frac{u^{\alpha}}{|u\cdot z|}$$

where

| | |
|---|---|
| Line of observation | $z^{\mu} = x^{\mu} - X^{\mu}(\tau_R)$ |
| Lightcone condition | $z^2 = (x - X(\tau_R))^2 = 0$ |
| 4-velocity of source | $u^{\mu} = \dot{X}^{\mu}(\tau_R)$ |
| Retarded causality | $\theta^{ret} = \theta\left(x^0 - X^0(\tau_R)\right)$ |

# Liénard-Wiechert Potential and Fields

## Field strengths

### Retarded fields

$$f^{\mu\nu}_{ret}(x,\tau) = -\frac{e}{4\pi}\varphi\left(\tau-\tau_R\right)\frac{\left(z^\mu u^\nu - z^\nu u^\mu\right)u^2}{\left(u\cdot z\right)^3} \sim \frac{1}{\mathbf{z}^2}$$

$$f^{5\mu}_{ret}(x,\tau) = \frac{ec_5}{4\pi}\varphi\left(\tau-\tau_R\right)\frac{z^\mu u^2 - u^\mu\left(u\cdot z\right)}{\left(u\cdot z\right)^3} \sim \frac{c_5}{\mathbf{z}^2}$$

### Radiation fields

$$f^{\mu\nu}_{rad}(x,\tau) = -\frac{e}{4\pi}\varphi(\tau-\tau_R)\left[\frac{\left(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu\right)\left(u\cdot z\right) - \left(z^\mu u^\nu - z^\nu u^\mu\right)\left(\dot{u}\cdot z\right)}{\left(u\cdot z\right)^3} + \frac{\epsilon\left(\tau-\tau_R\right)}{\lambda}\frac{z^\mu u^\nu - z^\nu u^\mu}{\left(u\cdot z\right)^2}\right] \sim \frac{1}{|\mathbf{z}|}$$

$$f^{5\mu}_{rad}(x,\tau) = -\frac{ec_5}{4\pi}\varphi\left(\tau-\tau_R\right)\left[\frac{\left(\dot{u}\cdot z\right)z^\mu}{\left(u\cdot z\right)^3} - \frac{\epsilon\left(\tau-\tau_R\right)}{\lambda}\frac{z^\mu - u^\mu\left(u\cdot z\right)}{\left(u\cdot z\right)^2}\right] \sim \frac{c_5}{|\mathbf{z}|}$$

# Smooth Transition as $c_5 \to 0$

## Field equations $\to$ Maxwell

### Homogeneous equations

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \qquad \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5}\partial_\tau f_{\mu\nu} = 0$$

- Satisfied identically (field defined as exact form $f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha$)
- $\partial_\tau f_{\mu\nu} = c_5 \left(\partial_\nu f_{5\mu} - \partial_\mu f_{5\nu}\right) \to 0 \quad \Rightarrow \quad f_{\mu\nu} \to$ static

### Inhomogeneous equations

$$\partial_\nu f^{\mu\nu} - \partial_\tau \left(\frac{1}{c_5} f^{5\mu}\right) = \frac{e}{c}\, j^\mu_\varphi \qquad \partial_\mu \left(\frac{1}{c_5} f^{5\mu}\right) = \frac{1}{c}\, e\rho_\varphi$$

- Liénard-Wiechert potentials $\Rightarrow \dfrac{1}{c_5} f^{5\mu} \xrightarrow[c_5 \to 0]{}$ finite
- Equilibrium conditions: $\rho_\varphi(x,\tau) \to 0$ and $\partial_\tau f^{5\mu} = 0$
- $\tau$-independent Maxwell equation $\partial_\nu f^{\mu\nu} = \dfrac{e}{c}\, j^\mu_\varphi$
- Free field $\partial_\mu f^{5\mu} = 0$

# Smooth Transition as $c_5 \to 0$

## Lorentz force $\to$ Maxwell

### From Liénard-Wiechert fields

$$M\ddot{x}^\mu = \frac{e_0}{c}\left[f^\mu{}_\nu(x,\tau)\dot{x}^\nu + f^{5\mu}(x,\tau)\dot{x}^5\right]$$

$$= \frac{e_0}{c}\frac{e}{4\pi}\varphi(\tau-\tau_R)\left[\mathcal{F}^\mu{}_\nu(x,\tau)\dot{x}^\nu + c_5^2\ \mathcal{F}^{5\mu}(x,\tau)\right]$$

where

$$\mathcal{F}^{\mu\nu} = \frac{(z^\mu u^\nu - z^\nu u^\mu)\,u^2}{(u\cdot z)^3} + \left[\frac{(z^\mu\dot{u}^\nu - z^\nu\dot{u}^\mu)(u\cdot z) - (z^\mu u^\nu - z^\nu u^\mu)(\dot{u}\cdot z)}{(u\cdot z)^3} + \frac{\epsilon(\tau-\tau_R)}{\lambda}\frac{z^\mu u^\nu - z^\nu u^\mu}{(u\cdot z)^2}\right]$$

$$\mathcal{F}^{5\mu} = \frac{z^\mu u^2 - u^\mu(u\cdot z)}{(u\cdot z)^3} - \frac{(\dot{u}\cdot z)\,z^\mu}{(u\cdot z)^3} + \frac{\epsilon(\tau-\tau_R)}{\lambda}\frac{z^\mu - u^\mu(u\cdot z)}{(u\cdot z)^2}$$

are independent of $c_5$.

Using $\varphi(\tau) = \dfrac{1}{2\alpha\lambda}e^{-|\tau|/\alpha\lambda}$ and $\alpha = \frac{1}{2}\left[1+\left(\frac{c_5}{c}\right)^2\right]$

$$M\ddot{x}^\mu = \frac{e^2}{4\pi c}\,e^{-|\tau-\tau_R|/\alpha\lambda}\,\frac{\mathcal{F}^\mu{}_\nu\dot{x}^\nu + c_5^2\ \mathcal{F}^{5\mu}}{1+(c_5/c)^2} \xrightarrow[c_5\to 0]{} \frac{e^2}{4\pi c}\,e^{-2|\tau-\tau_R|/\lambda}\ \mathcal{F}^\mu{}_\nu\dot{x}^\nu$$

# Toy Model for Mass Shift

## On-shell event enters dense region of charged particles

Uniformly propagating event

$$x(\tau) = u\tau = \left(u^0, \mathbf{u}\right) \qquad\qquad u^2 = -c^2$$

Dense region of charged particles induces small stochastic perturbation $X(\tau)$

$$x(\tau) = u\tau + X(\tau)$$

Typical distance $d$ between force centers $\longrightarrow$ roughly periodic perturbation

$$\text{characteristic period} = \frac{d}{|\mathbf{u}|} = \frac{\text{very short distance}}{\text{moderate velocity}} = \text{very short time}$$

$$\text{fundamental frequency} = \omega_0 = 2\pi\frac{|\mathbf{u}|}{d} = \text{very high frequency}$$

$$\text{amplitude} = |X^\mu(\tau)| \sim \alpha d$$

$$\text{macroscopic factor} = \alpha < 1$$

# Toy Model for Mass Shift

## Perturbed motion

Expand perturbation in Fourier series

$$X(\tau) = \mathrm{Re}\sum_n a_n \; e^{in\omega_0\tau}$$

Write four-vector coefficients as

$$a_n = \alpha d s_n = \alpha d \left(s_n^0, \mathbf{s}_n\right) = \alpha d \left(c s_n^t, \mathbf{s}_n\right)$$

where $s_n$ represent normalized Fourier series ($s_0^\mu \sim 1$)

Perturbed motion on microscopic scale $d$

$$X(\tau) = \alpha d \; \mathrm{Re}\sum_n s_n^\mu \; e^{in\omega_0\tau}$$

Perturbed velocity on macroscopic scale $\alpha\,|\mathbf{u}|$

$$\dot{x}^\mu(\tau) = u^\mu + \dot{X}^\mu(\tau) = u^\mu + \alpha d \; \mathrm{Re}\sum_n n\omega_0 \; s_n^\mu \; i e^{in\omega_0\tau}$$

$$= u^\mu + \alpha d \; \mathrm{Re}\sum_n n\left(2\pi\frac{|\mathbf{u}|}{d}\right) s_n^\mu \; i e^{in\omega_0\tau} = u^\mu + \alpha\,|\mathbf{u}| \;\; \mathrm{Re}\sum_n 2\pi n \; s_n^\mu \; i e^{in\omega_0\tau}$$

# Toy Model for Mass Shift

## Perturbed mass

Unperturbed on-shell mass

$$m = -\frac{M\dot{x}^2(\tau)}{c^2} = M$$

Perturbed mass

$$m = -\frac{M\dot{x}^2(\tau)}{c^2} = -\frac{M}{c^2}\left(u + \alpha\,|\mathbf{u}|\ \text{Re}\sum_n 2\pi n\ s_n\ ie^{in\omega_0\tau}\right)^2$$

$$= M\left(1 - \frac{2\alpha\,|\mathbf{u}|}{c^2}\text{Re}\sum_n 2\pi n\,(u\cdot s_n)\,ie^{in\omega_0\tau} + \frac{\alpha^2\mathbf{u}^2}{c^2}\text{Re}\sum_{n,m}(2\pi)^2\,nm\ s_n\cdot s_m\ e^{i(n+m)\omega_0\tau}\right)$$

In rest frame of unperturbed motion, neglecting $\alpha^2$

$$\frac{2\alpha\,|\mathbf{u}|}{c^2}2\pi n\ \ (u\cdot s_n) = \frac{4\pi\alpha\,|\mathbf{u}|\,n}{c^2}\ \ (c,\mathbf{0})\cdot\left(cs_n^t,\mathbf{s}_n\right) = -4\pi\alpha\,|\mathbf{u}|\ \ ns_n^t$$

$$m \longrightarrow m\left(1+\frac{\Delta m}{m}\right) \qquad \frac{\Delta m}{m} = 4\pi\alpha\,|\mathbf{u}|\,\text{Re}\sum_n n\ s_n^t\ \ ie^{in\omega_0\tau}$$

Larger mass shifts if $\alpha > 1 \ \Rightarrow\ \alpha^2$ becomes significant

# Self-Interaction for Mass Stability

## Terminal velocity

Mass exchange with electromagnetic field

$$\frac{d}{d\tau}(-\tfrac{1}{2}M\dot{x}^2) = \frac{e_0}{c}\, f^{5\mu}\dot{x}_\mu = \frac{e^2}{4\pi c}\, e^{-|\tau-\tau_R|/\alpha\lambda}\, \frac{c_5^2}{1+(c_5/c)^2}\, \mathcal{F}^{5\mu}\dot{x}_\mu$$

- Scaled by $c_5^2$
- Small but possibly significant
- Seek mechanism that damps off-shell mass
- Analogous to terminal velocity from friction

Toy model: $f^{5\mu} = \sigma\dot{x}^\mu$

$$\frac{d}{d\tau}(-\tfrac{1}{2}M\dot{x}^2) = \frac{e_0}{c}\,\sigma\dot{x}^\mu\dot{x}_\mu = -\frac{2e_0\sigma}{Mc}\left(-\tfrac{1}{2}M\dot{x}^2\right)$$

$$-\tfrac{1}{2}M\dot{x}^2(\tau) = -\tfrac{1}{2}M\dot{x}^2(0)\; e^{-(2e_0\sigma/Mc)\tau}$$

# Self-Interaction for Mass Stability

## Framework

- Arbitrarily moving event $X^\mu(\tau)$
- Produces current $j^\alpha_\varphi(x,\tau)$ and field $f^{\alpha\beta}(x,\tau)$
- At time $\tau^* > \tau$ event interacts with its own field
- In co-moving frame

$$X(\tau) = (ct(\tau), \mathbf{0}) \qquad \dot{X}(\tau) = (c\dot{t}(\tau), \mathbf{0})$$

- $G_{Maxwell} = 0$ on timelike separation

$$X(\tau^*) - X(\tau) = c(t(\tau^*) - t(\tau), \mathbf{0})$$

- Contribution from $G_{Correlation}$

$$a^\alpha(X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3} \int ds\ \dot{X}^\alpha(s) \left( \frac{1}{2} \frac{\theta(g(s))}{(g(s))^{3/2}} - \frac{\delta(g(s))}{(g(s))^{1/2}} \right) \theta^{ret}$$

$$c^2 g(s) = -\left( (X(\tau) - X(s))^2 + c_5^2 (\tau - s)^2 \right) = c^2 \left( (t(\tau^*) - t(s))^2 - \frac{c_5^2}{c^2} (\tau^* - s)^2 \right)$$

# Self-Interaction for Mass Stability

## On-shell motion

For the event $t(\tau^*) = \tau^* \;\Rightarrow\; g(s) = \left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2 \qquad \theta^{ret} = \theta(\tau^* - s)$

$$a(X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3}(c, \mathbf{0}, c_5)\int_{-\infty}^{\tau^*}\left(\frac{\theta\left(\left(1-\frac{c_5^2}{c^2}\right)(\tau^*-s)^2\right)}{2\left(\left(1-\frac{c_5^2}{c^2}\right)(\tau^*-s)^2\right)^{3/2}} - \frac{\delta\left(\left(1-\frac{c_5^2}{c^2}\right)(\tau^*-s)^2\right)}{\left(\left(1-\frac{c_5^2}{c^2}\right)(\tau^*-s)^2\right)^{1/2}}\right)$$

$$= \frac{ec_5(c, \mathbf{0}, c_5)}{2\pi^2 c^3\left(1-\frac{c_5^2}{c^2}\right)^{3/2}}\int_{-\infty}^{\tau^*} ds\left(\frac{1}{2(\tau^*-s)^3} - \frac{\delta(\tau^*-s)\,\theta(\tau^*-s)}{\left|(\tau^*-s)^2\right|}\right)$$

Evaluating

$$\int_{-\infty}^{\tau^*} ds\,\frac{1}{(\tau^*-s)^3} = \left.\frac{1}{2(\tau^*-s)^2}\right|_{-\infty}^{\tau^*} = \lim_{s\to\tau^*}\frac{1}{2(\tau^*-s)^2}$$

$$\int_{-\infty}^{\tau^*} ds\,\frac{\delta(\tau^*-s)\,\theta(\tau^*-s)}{(\tau^*-s)^2} = \lim_{s\to\tau^*}\frac{\theta(\tau^*-s)}{(\tau^*-s)^2} = \lim_{s\to\tau^*}\frac{\frac{1}{2}}{(\tau^*-s)^2}$$

Induced potential vanishes

$$a(X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3}(c, \mathbf{0}, c_5)\lim_{s\to\tau^*}\left(\frac{1}{2(\tau^*-s)^2} - \frac{\frac{1}{2}}{(\tau^*-s)^2}\right) = 0$$

# Self-Interaction for Mass Stability

## Arbitrary motion in co-moving frame

Since $\dot{X}(\tau) = (c\dot{t}(\tau), \mathbf{0})$

$$a^i = 0 \qquad \partial_i a^0 = \partial_i a^5 = 0 \qquad \Rightarrow \qquad f^{\mu\nu} = f^{5i} = 0$$

Field strength reduces to

$$f^{50} = \partial^5 a^0 - \partial^0 a^5 = g^{55} \frac{1}{c_5} \partial_{\tau^*} a^0 - g^{00} \frac{1}{c} \partial_t a^5 = \frac{1}{c_5} \partial_{\tau^*} a^0 + \frac{1}{c} \partial_t a^5$$

where $\partial_{\tau^*}$ acts only on explicit variable: $\partial_{\tau^*} \dot{X}^\alpha(s) = \partial_{\tau^*} t(\tau^*) = \partial_{\tau^*} \theta^{ret} = 0$

Working through derivatives of

$$a^\alpha(X(\tau^*), \tau^*) = \frac{e c_5}{2\pi^2 c^3} \int ds \; \dot{X}^\alpha(s) \left( \frac{1}{2} \frac{\theta(g(s))}{(g(s))^{3/2}} - \frac{\delta(g(s))}{(g(s))^{1/2}} \right) \theta^{ret}$$

Leads to

$$f^{50} = f^{50}_\theta + f^{50}_\delta + f^{50}_{\delta'}$$

$$= \text{terms containing } \theta(g(s)) + \text{terms containing } \delta(g(s))$$

$$+ \text{ terms containing } \partial_{\tau^*} \delta(g(s))$$

Field strength

$$f^{50} = f^{50}_{\theta} + f^{50}_{\delta} + f^{50}_{\delta'}$$

where

$$f^{50}_{\theta} = \frac{3e}{4\pi^2}\frac{c_5^2}{c^4}\int ds \, \frac{\theta\left(g(s)\right)}{\left(g(s)\right)^{5/2}} \, \theta^{ret} \, \Delta\left(\tau^*, s\right)$$

$$f^{50}_{\delta} = -\frac{e}{\pi^2}\frac{c_5^2}{c^4}\int ds \, \frac{\delta\left(g(s)\right)}{\left(g(s)\right)^{3/2}} \, \theta^{ret} \, \Delta\left(\tau^*, s\right)$$

$$f^{50}_{\delta'} = -\frac{e}{\pi^2}\frac{c_5^2}{c^4}\int ds \, \frac{\delta'\left(g(s)\right)}{\left(g(s)\right)^{1/2}} \, \theta^{ret} \, \Delta\left(\tau^*, s\right)$$

and

$$\Delta\left(\tau^*, s\right) = \dot{t}(s)(\tau^* - s) - \left(t\left(\tau^*\right) - t\left(s\right)\right)$$

$$g(s) = \left(t\left(\tau^*\right) - t\left(s\right)\right)^2 - \frac{c_5^2}{c^2}(\tau^* - s)^2$$

# Self-Interaction for Mass Stability

## Function $\Delta(\tau^*, s)$

Event at constant velocity

$$x^0(\tau) = u^0\tau \qquad \Rightarrow \qquad \Delta(\tau^*, s) = \frac{u^0}{c}(\tau^* - s) - \left(\frac{u^0}{c}\tau^* - \frac{u^0}{c}s\right) = 0$$

For any smooth $t(\tau)$

$$\begin{aligned} t(\tau^*) - t(s) &= t(s) + \dot{t}(s)(\tau^* - s) + \frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right) - t(s) \\ &= \dot{t}(s)(\tau^* - s) + \frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right) \end{aligned}$$

leading to

$$\Delta(\tau^*, s) = \dot{t}(s)(\tau^* - s) - (t(\tau^*) - t(s)) = -\frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right)$$

so that

$\Delta(\tau^*, s) \neq 0 \;\Rightarrow\;$ time coordinate accelerates in rest frame $\;\Rightarrow\;$ particle mass shift

# Self-Interaction for Mass Stability

## Mass jump

Small, sudden jump in mass at $\tau = 0$

$$t(\tau) = \begin{cases} \tau & , \quad \tau < 0 \\ (1+\beta)\,\tau & , \quad \tau > 0 \end{cases} \qquad \Rightarrow \qquad \dot{t}(\tau) = \begin{cases} 1 & , \quad \tau < 0 \\ 1+\beta & , \quad \tau > 0 \end{cases}$$

For $\tau^* < 0$

$$\theta^{ret} \;\Rightarrow\; s < 0 \quad \longrightarrow \quad \dot{t}(\tau^*) = \dot{t}(s) = 1 \quad \longrightarrow \quad \Delta(\tau^*, s) = 0$$

For $\tau^* > 0$

$$s > 0 \quad \longrightarrow \quad \dot{t}(\tau^*) = \dot{t}(s) = 1 + \beta \quad \longrightarrow \quad \Delta(\tau^*, s) = 0$$

$$s < 0 \quad \longrightarrow \quad \Delta(\tau^*, s) = \dot{t}(s)\,(\tau^* - s) - ((1+\beta)\,(\tau^*) - s) = -\beta\tau^*$$

Solving for

$$g(s^*) = (t\,(\tau^*) - t\,(s))^2 - \frac{c_5^2}{c^2}(\tau^* - s)^2 = ((1+\beta)\,\tau^* - s)^2 - \frac{c_5^2}{c^2}(\tau^* - s)^2 = 0$$

$$s^* = \left(1 + \frac{\beta}{1 - \frac{c_5}{c}}\right)\tau^* > \tau^* \quad \Rightarrow \quad g(s) > 0 \text{ on } s < 0 < \tau^* \quad \Rightarrow \quad f_\delta^{50} = f_{\delta'}^{50} = 0$$

# Self-Interaction for Mass Stability

## Field strength from mass jump

Since $\theta(g(s)) = 1$ for $s < \tau^*$ and $\Delta(\tau^*, s) = \begin{cases} -\beta\tau^* & , \quad \text{for } s < 0 \\ 0 & , \quad \text{for } s > 0 \end{cases}$

The field strength is

$$f^{50} = f_\theta^{50} = (-\beta\tau^*) \frac{3e}{4\pi^2} \frac{c_5^2}{c^4} \int_{-\infty}^0 ds \; \frac{1}{[g(s)]^{5/2}}$$

$$= (-\beta\tau^*) \frac{3e}{4\pi^2} \frac{c_5^2}{c^4} \int_{-\infty}^0 ds \; \frac{1}{\left[ ((1+\beta)\,\tau^* - s)^2 - \frac{c_5^2}{c^2} (\tau^* - s)^2 \right]^{5/2}}$$

$$= \frac{e}{4\pi^2} \frac{1}{c_5^2 \, (\beta\tau^*)^3} \; Q\left(\beta, \frac{c_5^2}{c^2}\right)$$

where $Q\left(\beta, \frac{c_5^2}{c^2}\right)$ is positive, dimensionless, finite for $c_5 < c$ and

$$Q\left(\beta, \frac{c_5^2}{c^2}\right) \xrightarrow[c_5 \to 0]{} 0$$

# Self-Interaction for Mass Stability

## Factor $Q$

$$Q\left(\beta, \frac{c_5^2}{c^2}\right) = \left[ 2\left(1 - \frac{c_5^2}{c^2}\right)^{3/2} \left( 1 - \frac{\left(1 - \frac{c_5^2}{c^2}\right)^{1/2} \left(1 + \frac{\beta}{\left(1 - \frac{c_5^2}{c^2}\right)}\right)}{\left[1 + \frac{2\beta}{1 - \frac{c_5^2}{c^2}} + \frac{\beta^2}{1 - \frac{c_5^2}{c^2}}\right]^{1/2}} \right) \right.$$

$$\left. + \frac{\beta^2 \ \frac{c_5^2}{c^2} \left(1 + \frac{c_5^2}{c^2} \frac{\beta}{1 - \frac{c_5^2}{c^2}}\right)}{\left(1 - \frac{c_5^2}{c^2}\right)^{1/2} \left[1 + \frac{2\beta}{1 - \frac{c_5^2}{c^2}} + \frac{\beta^2}{1 - \frac{c_5^2}{c^2}}\right]^{3/2}} \right]$$

# Self-Interaction for Mass Stability

## Lorentz force

Since $f^{\mu\nu} = 0$,

$$M\ddot{x}^{\mu} = e_0 f^{\mu\alpha}\dot{x}_{\alpha} = e_0 f^{\mu 5}\dot{x}_5 = -e_0 f^{5\mu}\dot{x}_5 = -g_{55} e_0 f^{5\mu}\dot{x}^5 = -e_0 c_5 f^{5\mu}$$

Self-interaction is

$$M\ddot{x}^0 = -c_5 e_0 f^{50} = \begin{cases} 0 & , \quad \tau^* < 0 \\ -\dfrac{\lambda e^2}{4\pi^2}\dfrac{1}{c_5\left(\beta\tau^*\right)^3}\, Q\left(\beta, \dfrac{c_5^2}{c^2}\right) & , \quad \tau^* > 0 \end{cases}$$

$$M\ddot{x}^i = -c_5 e_0 f^{5i}\dot{x}_i = 0$$

$$\frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^2\right) = e_0 f^{5\mu}\dot{x}_{\mu} = -e_0 c f^{50}\dot{t} = -\frac{\lambda e^2}{4\pi^2}\frac{c}{c_5^2\left(\beta\tau^*\right)^3}\, Q\left(\beta, \frac{c_5^2}{c^2}\right)\dot{t}$$

Emergent picture

Self-interaction $\longrightarrow$ force opposing mass exchange

Mass damps back to on-shell value

Force vanishes when $\dot{t} = 1$