Pre-Maxwell Electrodynamics

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Abstract

In the context of a covariant mechanics with Poincaré-invariant evolution parameter τ, Sa'ad, Horwitz, and Arshansky have argued that for the electromagnetic interaction to be well-posed, the local gauge function of the field should include dependence on τ, as well as on the spacetime coordinates. This requirement of full gauge covariance leads to a theory of five τ-dependent gauge compensation fields, which differs in significant aspects from conventional electrodynamics, but whose zero modes coincide with the Maxwell theory. The pre-Maxwell fields may exchange mass with charged particles, permitting pair annihilation even at the classical level. The total mass-energy-momentum tensor of the fields and particles is conserved. The Green's functions for the fields provide spacelike and timelike support for correlations, as well as lightlike propagation. A τ-integration of the fields — singling out the massless photons — recovers the standard Maxwell theory, which then has the character of an equilibrium limit of the underlying microscopic dynamics.

The pre-Maxwell theory also turns out to be the solution of the inverse problem in variational mechanics: it is shown to be the most general local gauge theory consistent with unconstrained commutation relations in four dimensions. Posed in this framework, the extension to n-dimensions, curved background space, and non-abelian gauge symmetry becomes straightforward.

1 Introduction

The 'proper time' equation which Feynman [1] used to describe a Klein-Gordon particle in an external electromagnetic field,

$$i\frac{\partial}{\partial s}\psi_s(x) = K\psi_s(x) = (p - eA)^2\psi_s(x) \ , \tag{1}$$

or the equivalent Heisenberg picture formalism used by Schwinger [2],

$$[x^\mu, \pi^\nu] = ig^{\mu\nu} \qquad\qquad [\pi^\mu, \pi^\nu] = ieF^{\mu\nu}$$

$$i[x^\mu, K] = -\frac{\partial x^\mu}{\partial s} \qquad\qquad i[\pi^\mu, K] = -\frac{\partial \pi^\mu}{\partial s} \ , \tag{2}$$

is associated with the five-dimensional conserved current

$$\partial_\mu j^\mu + \partial_s \rho = 0 \tag{3}$$

where

$$\rho = \Big|\psi_s(x)\Big|^2 \qquad j^\mu = -i\Big\{\psi^*(\partial^\mu - ieA^\mu)\psi - \psi(\partial^\mu + ieA^\mu)\psi^*\Big\} \ . \tag{4}$$

In his approach to 'proper time' quantum mechanics, Stueckelberg [3] regarded (3) as a true particle current, leading to the interpretation of $\Big|\psi_s(x)\Big|^2$ as a probability density. However, the four-vector current j^μ acquires a non-zero divergence if this probability density evolves under s, and so it may not be identified as the source of the s-independent Maxwell field $A^\mu(x)$. Nevertheless, as Steuckelberg was aware, assuming the boundary conditions $\rho \to 0$ pointwise as $s \to \pm\infty$, integration of (3) over s leads to

$$\partial_\mu J^\mu = 0 \qquad \text{where} \qquad J^\mu(x) = \int_{-\infty}^{\infty} ds \ j^\mu(x,s) \ , \tag{5}$$

suggesting a 'proper time' quantum mechanics in which particles interact via the s-independent Maxwell field $A^\mu(x)$, and the fields are in turn induced by the divergenceless particle current J^μ.

However, in the resulting dynamical picture, the fields which mediate particle interaction instantaneously at s, are induced by currents whose support covers the particle worldlines, past and future. There is no *a priori* assurance that the particles moving in these Maxwell fields will trace out precisely the worldlines which induce the fields responsible for their motion, and this theory may not be well-posed.

2 Local Gauge Invariance

In order to obtain a well-posed theory, Sa'ad, Horwitz, and Arshansky [4] have argued that the local gauge function of the field should include dependence on the evolution parameter as well as on the spacetime coordinates (see also [5]). This requirement of full gauge covariance leads to a theory of five gauge compensation fields, which differs in significant aspects from conventional electrodynamics, but whose zero modes coincide with the Maxwell theory.

Under local gauge transformations of the form

$$\psi(x,\tau) \rightarrow e^{ie_0\Lambda(x,\tau)}\psi(x,\tau) \tag{6}$$

the equation

$$(i\partial_\tau + e_0 a_5)\psi(x,\tau) = \frac{1}{2M}(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu)\psi(x,\tau) \tag{7}$$

is invariant, when the compensation fields transform as

$$a_\mu(x,\tau) \rightarrow a_\mu(x,\tau) + \partial_\mu \Lambda(x,\tau) \qquad a_5(x,\tau) \rightarrow a_5(x,\tau) + \partial_\tau \Lambda(x,\tau) \ , \tag{8}$$

where the potentials $a_\mu(x,\tau)$ and $a_5(x,\tau)$ must now also depend explicitly on the *world time* τ. This Schrödinger equation (7) leads (as for (1)) to the five dimensional conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = 0 \tag{9}$$

where

$$j^5 \equiv \rho = \left|\psi(x,\tau)\right|^2 \qquad j^\mu = \frac{-i}{2M}\left\{\psi^*(\partial^\mu - ie_0 a^\mu)\psi - \psi(\partial^\mu + ie_0 a^\mu)\psi^*\right\} \ , \tag{10}$$

and the interpretation of $\left|\psi(x,\tau)\right|^2$ as a probability density still holds. The current conservation law may be written as $\partial_\alpha j^\alpha = 0$, where the index convention is now

$$\lambda, \mu, \nu = 0,1,2,3 \qquad \text{and} \qquad \alpha, \beta, \gamma = 0,1,2,3,5 \tag{11}$$

and the parameter τ is formally designated x^5, so that $\partial_\tau = \partial_5$. In analogy with the non-relativistic case, the two-body action-at-a-distance potential in the Horwitz-Piron theory may be understood as the approximation

$$a_\mu(x,\tau) \longrightarrow 0 \qquad - e_0 a_5(x,\tau) \longrightarrow V(x) \ . \tag{12}$$

Within this framework, solutions have been found for the generalizations of the standard central force problem, including potential scattering [6] and bound states [7, 8] with radiative transitions [9] and Zeeman spectra [10]. In these solutions, scalar fields whose mass is continuous and unrestricted, form bound states whose discrete mass spectra split in an external magnetic field with multiplicity two.

3 Equations of Motion

3.1 The Field Equations

The Schrödinger equation (7) may be derived by variation of the action

$$\mathrm{S} = \int d^4x d\tau \left\{ \psi^*(i\partial_\tau + e_0 a_5)\psi - \frac{1}{2M}\psi^*(p_\mu - e_0 a_\mu)(p^\mu - e_0 a^\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\} \tag{13}$$

in which Sa'ad, et. al. have included a kinetic term for the fields, formed from the gauge invariant quantity

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \ . \tag{14}$$

In the kinetic term, the $\beta = 5$ index in $f_{\mu 5} = \partial_\mu a_5 - \partial_\tau a_\mu$ must be formally raised, and Sa'ad, et. al. argue that this term suggests the higher symmetry O(4,1) or O(3,2), corresponding to $g^{55} = \sigma = \pm 1$ in the free field metric

$$g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1, \sigma) \ . \tag{15}$$

Varying the action (13) with respect to the gauge fields, the equations of motion are found to be

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \qquad\qquad \epsilon^{\alpha\beta\gamma\delta\epsilon} \partial_\alpha f_{\beta\gamma} = 0 \tag{16}$$

where j^α is given by (10). As we show below, λ and e_0 are dimensional constants; one identifies e_0/λ as the dimensionless Maxwell charge e. In four-vector component form, (16) becomes

$$\partial_\nu \ f^{\mu\nu} - \partial_\tau \ f^{5\mu} \ = \ e j^\mu \qquad\qquad \partial_\mu \ f^{5\mu} = e\rho. \tag{17}$$

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} \ = \ 0 \qquad\qquad \partial_\mu f_{5\nu} - \partial_\nu f_{5\mu} + \partial_\tau f_{\mu\nu} = 0 \ , \tag{18}$$

which may be seen as a four-dimensional analog of the three-vector Maxwell equations in the usual form. The noncovariant form of the field equations [11] are defined through

$$e_i = f^{0i} \qquad h_i = \frac{1}{2}\epsilon_{ijk} f^{jk} \qquad \epsilon^i = f^{4i} \qquad \epsilon^0 = f^{40}. \tag{19}$$

Plane wave solutions for the source free case, where the Fourier transform is defined by

$$f(x,\tau) = \int d^4k d\kappa e^{i(k\cdot x + \sigma\kappa\tau)} f(k,\kappa) \ , \tag{20}$$

are

$$\mathbf{e} = \mathbf{e}_{\perp} + \sigma(\frac{\kappa}{k^0})\epsilon_{\|} \qquad \mathbf{h} = \frac{1}{k^0}\mathbf{k} \times \mathbf{e}_{\perp} \tag{21}$$

$$\epsilon = \epsilon_{\|} + (\frac{\kappa}{k^0})\mathbf{e}_{\perp} \qquad \epsilon^0 = \frac{1}{k^0}\mathbf{k} \cdot \epsilon_{\|} \tag{22}$$

$$\mathbf{k}^2 - (k^0)^2 + \sigma\kappa^2 = 0 \ , \tag{23}$$

where the independent field vectors are $\mathbf{e}_{\perp}$ (normal to $\mathbf{k}$) and $\epsilon_{\|}$ (parallel to $\mathbf{k}$). As in the Maxwell case, the premagnetic field $\mathbf{h}$ is normal to the propagation vector $\mathbf{k}$; the two pre-electric fields $\mathbf{e}$, ϵ, are in a plane normal to $\mathbf{h}$, but need not be normal to $\mathbf{k}$. We remark that integration of (20) over τ (or, through the Riemann- Lebesgue lemma, in the limit $\tau \to \infty$) selects the $\kappa = 0$ component; in this case, equations (21) – (23) reduce to the usual Maxwell form (for which $\mathbf{e}$ and $\mathbf{h}$ are perpendicular to $\mathbf{k}$ and to each other, with $\mathbf{e}^2 = \mathbf{h}^2$), and ϵ becomes *parallel* to $\mathbf{k}$, $k^0 = |\mathbf{k}|$ and $\epsilon^0 = |\epsilon|$.

3.2 The Classical Lorentz Force

From (7) we can write the classical Hamiltonian [11] as

$$K = \frac{1}{2M}(p^\mu - e_0 a^\mu)(p_\mu - e_0 a_\mu) - e_0 a_5 \tag{24}$$

and using the classical Hamilton equations [12]

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} \tag{25}$$

we write the classical Lagrangian [11],

$$L = \dot{x}^\mu p_\mu - K = \frac{1}{2}M\dot{x}^\mu\dot{x}_\mu + e_0\dot{x}^\mu a_\mu + e_0 a_5 = \frac{1}{2}M\dot{x}^\mu\dot{x}_\mu + e_0\dot{x}^\alpha a_\alpha \ . \tag{26}$$

Varying the Lagrangian with respect to x^μ, we obtain the classical Lorentz force [11]

$$M \, \ddot{x}^\mu = e_0 \; f^\mu{}_\alpha(x,\tau) \, \dot{x}^\alpha \tag{27}$$

where $f_{\alpha\beta}$ is given by the gauge invariant quantity given in (14). We observe that the four equations (27) also imply [11]

$$\frac{d}{d\tau}(\frac{1}{2}M\dot{x}^2) = M\dot{x}^\mu\ddot{x}_\mu = e_0\dot{x}^\mu(f_{\mu 5} + f_{\mu\nu}\dot{x}^\nu) = e_0\dot{x}^\mu f_{\mu 5} \tag{28}$$

from which it follows that mass need not be conserved in this theory, even at the classical level, and that pair annihilation is classically permitted [3]. Nevertheless, it was shown in reference [11], by a study of the energy-momentum-mass tensor for the classical motions, that the total energy, momentum, and mass of the particles plus fields are conserved. We may conclude from (28) that the conditions for the dynamical (as opposed to asymptotic) conservation of $\dot{x}^2$ = constant, are $f_{5\mu} = 0$ and $\partial_\tau f^{\mu\nu} = 0$. Thus, we see that the most general interaction which preserves the proper time constraint is of conventional Maxwell type, as employed by Schwinger [2] in his original use of the proper time method.

The sourceless gauge field equations inherit the formal five dimensional symmetry of the free gauge field Lagrangian, while the *physical* Lorentz covariance of the matter currents breaks the O(4,1) or O(3,2) symmetry of the free fields to O(3,1). Nevertheless, the wave equation associated with the fields is [4]

$$\partial_\alpha \partial^\alpha f^{\beta\gamma} = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) f^{\beta\gamma} = (\partial_\mu \partial^\mu + \sigma\, \partial_\tau^2) f^{\beta\gamma} = -e(\partial^\beta j^\gamma - \partial^\gamma j^\beta) \ , \qquad (29)$$

and the causal properties of the (free) Green's functions for the operator on the left hand side of (29) reflect the higher symmetry. The dependence of the wave equation on the signature σ of ∂_τ implies that these causal properties will be different for the symmetry groups O(3,2) and O(4,1). In addition to lightlike propagation, the Green's functions [13] support spacelike correlations in the O(3,2) case, and timelike correlations in the O(4,1) case; the timelike and spacelike correlations similarly do not carry information in the usual sense.

As in (5), when $j^5 \to 0$ pointwise, as $\tau \to \pm\infty$, integration of (10) over τ, leads to the divergenceless current

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau \; j^\mu(x,\tau) \qquad (30)$$

which we may identify as the source of the Maxwell field. Thus, under the boundary conditions $f^{5\mu} \to 0$ pointwise in x as $\tau \to \pm\infty$, integration of (16) over τ recovers the Maxwell equations, in the form,

$$\partial_\nu F^{\mu\nu} = e J^\mu \qquad\qquad \epsilon^{\mu\nu\rho\lambda} \partial_\mu F_{\nu\rho} = 0 \qquad (31)$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \; f^{\mu\nu}(x,\tau) \qquad \text{and} \qquad A^\mu(x) = \int_{-\infty}^{\infty} d\tau \; a^\mu(x,\tau) \qquad (32)$$

and so $a^\alpha(x,\tau)$ has been called the pre-Maxwell field. It follows from (32) that e_0 and λ have dimensions of length.

4 Generalizations

Using results of Hojman and Shepley [14] in the inverse problem of the calculus of variations, it may be shown that the commutation relations in d-dimensions

$$[x^\mu, x^\nu] = 0, \qquad m\,[x^\mu, \dot{x}^\nu] = -i\hbar\, g^{\mu\nu}(x), \tag{33}$$

are sufficient to establish the self-adjointness of the differential equations

$$m\,\ddot{x}^\mu = F^\mu(\tau, x, \dot{x}). \tag{34}$$

Given self-adjointness, it may be shown [15] that the most general admissible form for (34) is

$$m\frac{D\dot{x}^\mu}{D\tau} = m[\ddot{x}^\mu + \Gamma^{\mu\lambda\nu}\dot{x}_\lambda\dot{x}_\nu] = f^\mu{}_d(\tau, x) + f^{\mu\nu}(\tau, x)\,\dot{x}_\nu \tag{35}$$

where $\Gamma^{\nu\lambda\mu}(x)$ is the usual Levi-Cevita connection and the fields satisfy

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \qquad \partial_\mu f_{\nu d} + \partial_\nu f_{d\mu} + \partial_\tau\, f_{\mu\nu} = 0 \tag{36}$$

$$\mathcal{D}_\nu\, f^{\mu\nu} + \partial_\tau\, f^{\mu d} = j^\mu \qquad \mathcal{D}_\mu\, f^{d\mu} = \rho \tag{37}$$

with covariant derivative $\mathcal{D}_\mu$. It also follows that this system of differential equations generalizes to curved space the Lagrangian formulation of the classical pre-Maxwell theory given in (26). A generalization to classical non-Abelian symmetry (based on the formulation of classical SU(2) symmetry by Wong [16]) may also be given in this framework [15].

5 Interpretation of the Formalism

In the pre-Maxwell theory, interactions take place between events in spacetime rather than between worldlines. Each event, occurring at τ, induces a current density in spacetime which, for free particles (and hence asymptotically), disperses for large τ, and the continuity equation (9) states that these current densities evolve as the event density j^5 progresses through spacetime as a function of τ. As noted above, if $j^5 \to 0$ as $|\tau| \to \infty$, then j^μ may be identified with the Maxwell current when integrated over τ. This integration has been called concatenation [17] and provides the link between the event along a worldline and the notion of a particle, whose support is the entire worldline. Concatenation is evidently related to the Schwinger's parametric integration of the Green's function,

$$\begin{aligned} G(x,x') &= <x|\frac{1}{(p-eA)^2+m^2-i\epsilon}|x'> \\ &= i\int_0^\infty ds\; e^{-i(m^2-i\epsilon)s}\; <x|e^{-i(p-eA)^2 s}|x'> \\ &= \int_{-\infty}^\infty ds\; e^{-i(m^2-i\epsilon)s}\; G(x,x';s) \end{aligned} \tag{38}$$

which Feynman argues, chooses the mass m eigenstate for fixed mass Klein-Gordon particle. Following Feynman's interpretation, concatenation imposes the requirement that the Maxwell electromagnetic field be the zero mode with respect to the conjugate mass variable. The Maxwell theory thus has the character of an *equilibrium limit* of the microscopic pre-Maxwell theory. Shnerb and Horwitz [18] have given an interpretation of the dimensional constant λ as a coherence length, beyond which the off-shell photons decouple from matter. In this sense, the Maxwell theory is a correlation limit of the pre-Maxwell theory, in which it is properly contained. This interpretation is made more concrete in a discussion of classical Coulomb scattering [19]. Frastai and Horwitz [20] have shown that close to this limit, the strongest singularities of one-loop diagrams are regularized in a manner similar to that of Pauli and Villars [21].

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